

MORE ON SIGNED GRAPHS WITH AT MOST THREE EIGENVALUES

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Abstract

We consider signed graphs with just 2 or 3 distinct eigenvalues, in particular (i) those with at least one simple eigenvalue, and (ii) those with vertex-deleted subgraphs which themselves have at most 3 distinct eigenvalues. We also construct new examples using weighing matrices and symmetric 3-class association schemes.

Keywords: adjacency matrix, simple eigenvalue, strongly regular signed graph, vertex-deleted subgraph, weighing matrix, association scheme.

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1. INTRODUCTION

A *signed graph* \dot{G} is defined to be a pair (G, σ) , in which $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma: E \rightarrow \{1, -1\}$ is the *sign function*, also known as the *signature*. The *order* of a signed graph, denoted by n , is the number of its vertices. The edge set of \dot{G} consists of the subsets of positive and negative edges. We interpret an unsigned graph as a signed graph in which all edges are positive.

The *adjacency matrix* $A_{\dot{G}}$ of \dot{G} is the $n \times n$ $(0, 1, -1)$ -matrix which is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. By the *spectrum* of \dot{G} , we mean the spectrum of $A_{\dot{G}}$. An eigenvalue of \dot{G} is called a *main* eigenvalue if the corresponding eigenspace is not orthogonal to the all-1 vector. Throughout the paper, by the statement ' \dot{G} has k eigenvalues' we mean that \dot{G} has exactly k distinct eigenvalues.

In Section 2 we give some terminology and notation, and prove some auxiliary results. The problem of classifying graphs with a comparatively small number of eigenvalues has attracted a great deal of attention in the last 70 years; some recent results can be found in [4, 5, 7, 16, 17]. In [14] we considered regular and non-regular signed graphs with at most 3 eigenvalues; here we continue this research and pay more attention to non-regular signed graphs. In Section 3, we consider connected signed graphs with 3 eigenvalues at least one of which is simple. The number of simple eigenvalues governs our investigation in Section 4 of vertex-deleted subgraphs which themselves have 3 eigenvalues. In Section 5 we construct some signed graphs with 2 or 3 eigenvalues using weighing matrices or symmetric 3-class association schemes, and note the implications for vertex-deleted subgraphs.

2. PRELIMINARIES

We write I , O , J , $\mathbf{0}$ and \mathbf{j} for an identity matrix, an all-0 matrix, an all-1 matrix, an all-0 vector and an all-1 vector, respectively. Subscripts indicate size as necessary.

A signed graph \dot{G} is said to be connected, complete, regular or bipartite if the same holds for its underlying graph. The degree of a vertex in \dot{G} is the degree of the same vertex in G . The *net-degree* of a vertex i , denoted by d_i^\pm , is the difference between the numbers of positive and negative edges incident with i . A signed graph in which vertex net-degrees are equal is called *net-regular*. Similarly, a *net-biregular* signed graph is a signed graph which has 2 distinct net-degrees. It is known that \dot{G} is net-regular if and only if \mathbf{j} is an eigenvector of \dot{G} , and then \mathbf{j} belongs to the eigenspace of the net-degree [20].

A signed graph is said to be *homogeneous* if all its edges have the same sign

(in particular, if its edge set is empty). Otherwise, it is said to be *inhomogeneous*. The *negation* $-\dot{G}$ is obtained by reversing the sign of every edge of \dot{G} .

We say that signed graphs \dot{G} and \dot{H} are *isomorphic* if there is a permutation matrix P such that $A_{\dot{H}} = P^{-1}A_{\dot{G}}P$. In this case we write $\dot{G} \cong \dot{H}$. We say that \dot{G} and \dot{H} are *switching equivalent* if there is a vertex subset $S \subseteq V(\dot{G})$, such that \dot{H} is obtained by reversing the sign of every edge with one vertex in S and the other in $V(\dot{G}) \setminus S$.

If the vertex labelling is transferred from the underlying graph common to \dot{G} and \dot{H} , then \dot{G} and \dot{H} are switching equivalent if and only if there is a diagonal matrix D with ± 1 on the diagonal such that $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$. Clearly, isomorphism and switching equivalence preserve the spectrum.

An *equitable partition* of a signed graph \dot{G} is a partition of the vertex set $V(\dot{G})$ into non-empty cells C_1, C_2, \dots, C_s , such that each cell induces a net-regular signed graph and for $1 \leq i < j \leq s$ the edges between C_i and C_j induce a net-biregular or net-regular signed graph, in which vertices from each of C_i, C_j are equal in net-degree.

We say that a signed graph \dot{G} is *strongly regular* (for short, \dot{G} is a *SRSG*) with parameters r, a, b, c if the entries of $A_{\dot{G}}^2$ satisfy

$$a_{ij}^{(2)} = \begin{cases} r & \text{if } i = j, \\ a & \text{if } i \overset{+}{\sim} j, \\ b & \text{if } i \overset{-}{\sim} j, \\ c & \text{if } i \not\sim j \text{ and } i \neq j. \end{cases}$$

Note that $a_{ij}^{(2)}$ is the difference between the numbers of positive and negative i - j walks of length 2 in \dot{G} . Accordingly, this definition generalizes the definition of strongly regular graphs. We mostly deal with SRSGs in Subsection 5.2.

In the forthcoming sections we frequently use the following result.

Proposition 1 [14]. *A connected signed graph \dot{G} has exactly one positive eigenvalue if and only if \dot{G} is switching equivalent to a non-trivial complete multipartite graph. If \dot{G} has exactly one non-negative eigenvalue, then \dot{G} is switching equivalent to a complete graph.*

We now transfer the following two results from the domain of unsigned graphs.

Proposition 2. *If A is a real symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ such that λ_1 is a simple eigenvalue, then*

$$\prod_{i=2}^k (A - \lambda_i I) = \left(\prod_{i=2}^k (\lambda_1 - \lambda_i) \right) \mathbf{x}\mathbf{x}^T,$$

where \mathbf{x} is a unit eigenvector associated with λ_1 . For $k = 3$, there exists an eigenvector \mathbf{a} for λ_1 , such that $(A - \lambda_2 I)(A - \lambda_3 I) = p\mathbf{a}\mathbf{a}^\top$, where

$$p = \begin{cases} 1 & \text{if } \lambda_1 \notin (\lambda_2, \lambda_3), \\ -1 & \text{if } \lambda_1 \in (\lambda_2, \lambda_3). \end{cases}$$

Proof. Considering the spectral decomposition of A , we see that there exists an orthogonal matrix X such that

$$\prod_{i=2}^k (A - \lambda_i I) = X \begin{pmatrix} \prod_{i=2}^k (\lambda_1 - \lambda_i) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} X^\top = \left(\prod_{i=2}^k (\lambda_1 - \lambda_i) \right) \mathbf{x}\mathbf{x}^\top,$$

where \mathbf{x} is a unit eigenvector of $\prod_{i=2}^k (A - \lambda_i I)$ afforded by $\prod_{i=2}^k (\lambda_1 - \lambda_i)$. The result follows since $A\mathbf{x} = \lambda_1\mathbf{x}$.

For $k = 3$, by taking $\mathbf{a} = \sqrt{p(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}\mathbf{x}$, we arrive at the desired result. ■

The previous theorem is a slight extension of the result in which A is the adjacency matrix of an unsigned graph, $k = 3$ and λ_1 is the largest eigenvalue [4, 7, 16]. Our formulation is more general in order to embrace signed graphs, with the possibility that λ_1 is not the largest eigenvalue.

Proposition 3. *Let \dot{G} be obtained from a signed graph \dot{H} of order n by adding a new vertex whose neighbourhood in \dot{H} is determined by the characteristic $(0, 1, -1)$ -vector \mathbf{r} . The characteristic polynomial of \dot{G} is given by*

$$(1) \quad P_{\dot{G}}(x) = P_{\dot{H}}(x) \left(x - \sum_{i=1}^m \frac{\|Q_i \mathbf{r}\|^2}{x - \mu_i} \right),$$

where $\mu_1, \mu_2, \dots, \mu_m$ are the distinct eigenvalues of \dot{H} and Q_1, Q_2, \dots, Q_m are the matrices of the orthogonal projections of \mathbb{R}^n onto the eigenspaces of \dot{H} with respect to the canonical basis.

Proof. Using the Schur matrix decomposition in conjunction with the known identity $\text{adj}(xI - A_{\dot{H}}) = \det(xI - A_{\dot{H}})(xI - A_{\dot{H}})^{-1}$, we obtain

$$\begin{aligned} P_{\dot{G}}(x) &= \det \begin{pmatrix} x & -\mathbf{r}^\top \\ -\mathbf{r} & xI - A_{\dot{H}} \end{pmatrix} = xP_{\dot{H}}(x) - \mathbf{r}^\top \text{adj}(xI - A_{\dot{H}}) \mathbf{r} \\ &= P_{\dot{H}}(x) (x - \mathbf{r}^\top (xI - A_{\dot{H}})^{-1} \mathbf{r}). \end{aligned}$$

Since $(xI - A_{\dot{H}})^{-1}$ has spectral decomposition $\sum_{i=1}^m \frac{1}{x - \mu_i} Q_i$, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left(x - \mathbf{r}^\top \left(\sum_{i=1}^m \frac{1}{x - \mu_i} Q_i \right) \mathbf{r} \right).$$

Now (1) follows since $\mathbf{r}^\top Q_i \mathbf{r} = \mathbf{r}^\top Q_i Q_i \mathbf{r} = \mathbf{r}^\top Q_i^\top Q_i \mathbf{r} = (Q_i \mathbf{r})^\top Q_i \mathbf{r} = \|Q_i \mathbf{r}\|^2$. ■

The ‘unsigned’ version of the previous theorem is well-known, see [6, Theorem 2.2.8]. The cone over a signed graph \dot{G} is obtained by adding a vertex v along with positive edges between v and every vertex of \dot{G} . We denote this cone by $K_1 \nabla \dot{G}$. The following result is a direct consequence of the previous one.

Corollary 4. *The cone over \dot{H} has the characteristic polynomial*

$$P_{K_1 \nabla \dot{H}}(x) = P_{\dot{H}}(x) \left(x - \sum_{i=1}^m \frac{n \beta_i^2}{x - \mu_i} \right),$$

where $\mu_1, \mu_2, \dots, \mu_m$ are distinct eigenvalues of \dot{H} and $\beta_1, \beta_2, \dots, \beta_m$ are the corresponding main angles defined by $\beta_i = \|Q_i \mathbf{j}\| / \sqrt{n}$.

3. SIGNED GRAPHS WITH 3 EIGENVALUES, AT LEAST ONE OF WHICH IS SIMPLE

In this section we give some characterizations of signed graphs described in the section title. We start with the following lemma.

Lemma 5. *If \dot{G} is a connected signed graph with 3 eigenvalues such that at least 2 of them are simple, then \dot{G} is switching equivalent to a complete bipartite graph.*

Proof. If every eigenvalue of \dot{G} is simple, then \dot{G} is switching equivalent to (the complete bipartite graph) $K_{1,2}$. Now suppose that λ is the unique non-simple eigenvalue. If $\lambda = 0$ then \dot{G} has exactly one positive eigenvalue, hence is switching equivalent to a complete multipartite graph by Proposition 2.1. Moreover, since its spectrum has the form $[-\rho, 0^{n-2}, \rho]$, \dot{G} is switching equivalent to a complete bipartite graph [6, p. 47].

If $\lambda \neq 0$ then \dot{G} has a connected subgraph without λ as an eigenvalue, namely K_1 . Since the eigenspace of λ has codimension 2, K_1 can be extended to a connected induced subgraph \dot{H} of order 2 without λ as an eigenvalue (see [6, Theorem 5.1.6], which can be extended to the framework of signed graphs with slight modifications in the proof). Since $\dot{H} \cong \pm K_2$ we have $\lambda \notin \{1, -1\}$. Since also $\lambda \neq 0$, we know from [13, Theorem 3.3] that \dot{G} has at most 4 vertices, and this case is resolved by inspection. ■

Now we consider signed graphs with 3 eigenvalues, exactly one of which is simple. Accordingly, we assume that a connected signed graph \dot{G} has spectrum $[\rho, \mu^m, \lambda^l]$, with $m, l \geq 2$ and $\mu > \lambda$. By Proposition 2, there is a non-zero vector $\mathbf{a} = (a_1, a_2, \dots, a_n)^\top$ such that $A\mathbf{a} = \rho\mathbf{a}$ and

$$(2) \quad (A - \mu I)(A - \lambda I) = p\mathbf{a}\mathbf{a}^\top,$$

where $A = A_{\dot{G}}$, $p = -1$ if $\mu > \rho > \lambda$, and $p = 1$ otherwise. By equating the diagonal entries of both sides, we get

$$(3) \quad d_i = pa_i^2 - \mu\lambda,$$

where d_i is degree of the vertex i .

Lemma 6. *If \dot{G} is a connected net-regular signed graph with spectrum $[\rho, \mu^m, \lambda^l]$, where ρ is its net-degree, then \dot{G} is regular.*

Proof. Every eigenvector afforded by ρ is constant, which, by (3), means that \dot{G} is regular. ■

We note in passing that, by [8], the signed graph \dot{G} mentioned in the previous result is strongly regular. Moreover, we have the following result, which will be used in our last section.

Lemma 7. *A connected inhomogeneous non-complete regular signed graph \dot{G} is net-regular with spectrum $[\rho, \mu^m, \lambda^l]$ (ρ being the net-degree) if and only if \dot{G} is strongly regular and its parameters satisfy $a + b = 2c \neq 0$.*

Proof. Let \dot{G} have spectrum as in the statement of the lemma. Since every eigenvector afforded by ρ is constant, by (2) we have

$$A^2 = (\mu + \lambda)A - \mu\lambda I + kJ, \quad \text{for some } k \neq 0.$$

Comparing the entries of the left and the right hand side, we conclude that \dot{G} is strongly regular with $a + b = 2k$ and $c = k \neq 0$. The converse follows directly from [8, Theorem 4.2]. ■

Next we deal with the case in which an eigenvalue other than ρ is the only non-main eigenvalue.

Theorem 8. *If \dot{G} is a connected signed graph with spectrum $[\rho, \mu^m, \lambda^l]$ ($m, l \geq 2$) such that only λ is non-main, then there is a non-zero constant α such that*

$$(4) \quad d_i = \alpha(d_i^\pm - \mu)^2 - \mu\lambda,$$

where d_i and d_i^\pm are the degree and the net-degree of the vertex i , respectively.

In particular, if \dot{G} is regular then it is net-biregular, the sum of the corresponding net-degrees is 2μ and \dot{G} is switching equivalent to a net-regular signed graph. If \dot{G} is biregular then it is switching equivalent to a net-biregular signed graph.

Proof. Observe first that \dot{G} is not net-regular, since it has more than one main eigenvalue. We retain the notation introduced in Lemma 6. Since ρ and μ are main, we have

$$(5) \quad (A - \rho I)(A - \mu I)\mathbf{j} = \mathbf{0},$$

as proved in [19] on the basis of the result for unsigned graphs which can be found in [15]. It follows that $A^2\mathbf{j} \in \text{span}\langle \mathbf{d}, \mathbf{j} \rangle$, where $A\mathbf{j} = \mathbf{d} = (d_1^\pm, d_2^\pm, \dots, d_n^\pm)^\top$. Moreover, since $\mathbf{a}^\top\mathbf{j} \neq 0$, (2) shows that $\mathbf{a} \in \text{span}\langle \mathbf{d}, \mathbf{j} \rangle$. Hence, we may write $\mathbf{a} = r\mathbf{d} + s\mathbf{j}$, where $r \neq 0$ as \dot{G} is not net-regular.

By (5), we have $A^2\mathbf{j} = (\rho + \mu)A\mathbf{j} - \rho\mu\mathbf{j}$, which together with (2), gives

$$(\rho - \lambda)\mathbf{d} - \mu(\rho - \lambda)\mathbf{j} = p\mathbf{a}(\mathbf{a}^\top\mathbf{j}) = p(\mathbf{a}^\top\mathbf{j})(r\mathbf{d} + s\mathbf{j}).$$

By equating the coefficients of \mathbf{d} and \mathbf{j} , we find that $s = -\mu r$, and so

$$(6) \quad \mathbf{a} = r(\mathbf{d} - \mu\mathbf{j}).$$

Using (3), we obtain $d_i = pr^2(d_i^\pm - \mu)^2 - \mu\lambda$, and by setting $\alpha = pr^2$, we arrive at (4).

Now, if \dot{G} is d -regular, then $d = \alpha(d_i^\pm - \mu)^2 - \mu\lambda$ for every vertex i . Evidently, this equation has 2 solutions in d_i^\pm and since \dot{G} is not net-regular, both solutions appear as net-degrees; hence, \dot{G} is net-biregular. The sum of the corresponding net-degrees follows from the previous equation for d . Lastly, from (3) we see that the coordinates of \mathbf{a} are equal in absolute value. If D is the diagonal matrix of ± 1 s with 1 in the i th position precisely when a_i is positive, then $D^{-1}AD$ is the adjacency matrix of a switching equivalent signed graph, say \dot{H} . Moreover, $D\mathbf{a}$ is a constant eigenvector associated with ρ in \dot{H} , which means that \dot{H} is net-regular.

Finally, suppose that \dot{G} is biregular with degrees d_1 and d_2 , and assume that \dot{G} is not net-biregular. Then, for at least one j ($j \in \{1, 2\}$), there are vertices of degree d_j which differ in net-degree. By (6), the corresponding coordinates of \mathbf{a} are different, while by (3), they are equal in absolute value. Using D formed exactly as before, we obtain a signed graph \dot{H} for which we have $A_{\dot{H}}D\mathbf{a} = \rho D\mathbf{a}$ where $D\mathbf{a}$ has 2 different coordinates. By (6), this means that \dot{H} has 2 net-degrees, and so it is net-biregular. ■

Of course, there is an analogous statement with μ in the role of the unique non-main eigenvalue. Here is a closer description of \dot{G} being net-biregular.

Corollary 9. *If the signed graph \dot{G} of Theorem 8 is net-biregular, then \dot{G} is biregular and its net-degrees determine an equitable vertex bipartition.*

Proof. From (4) we see that \dot{G} must be biregular. In addition, since \dot{G} is non-regular, its vertices are equal in degree if and only if they are equal in net-degree. By (6), the eigenvector \mathbf{a} of (2) has 2 different coordinates, say a_u and a_w , which correspond to different net-degrees and determine the vertex set partition $V = U \dot{\cup} W$. It remains to show that this partition is equitable. For $v \in V$, let d_{vu}^\pm and d_{vw}^\pm denote its net-degree in U and W , respectively. Then we also have $d_v^\pm = d_{vu}^\pm + d_{vw}^\pm$. If $v \in U$, since \mathbf{a} is associated with ρ , we have $d_{vu}^\pm a_u + d_{vw}^\pm a_w = \rho a_u$, i.e., $d_{vu}^\pm a_u + (d_v^\pm - d_{vu}^\pm) a_w = \rho a_u$ and $(d_v^\pm - d_{vu}^\pm) a_u + d_{vw}^\pm a_w = \rho a_u$. The last two equalities lead to

$$d_{vu}^\pm = \frac{\rho a_u - d_v^\pm a_w}{a_u - a_w} \quad \text{and} \quad d_{vw}^\pm = a_u \frac{d_v^\pm - \rho}{a_u - a_w}.$$

In a very similar way, we obtain

$$d_{vu}^\pm = a_w \frac{d_v^\pm - \rho}{a_w - a_u} \quad \text{and} \quad d_{vw}^\pm = \frac{\rho a_w - d_v^\pm a_u}{a_w - a_u},$$

for $v \in W$. In other words, the net-degrees determine an equitable vertex bipartition. ■

It is not difficult to construct some examples. For instance, by making a switch with respect to 3 mutually adjacent vertices of the Paley graph with 9 vertices, we obtain a regular and net-biregular signed graph with spectrum $[4, 1^4, (-2)^4]$. Also, the cone over the complete bipartite signed graph $\dot{K}_{4,4}$, in which negative edges form a perfect matching, is biregular and net-biregular, while its spectrum is $[4, 2^3, (-2)^5]$.

4. VERTEX-DELETED SUBGRAPHS WITH 3 EIGENVALUES

In this section we consider the question of whether a vertex-deleted subgraph of a connected signed graph \dot{G} with 3 eigenvalues also has 3 eigenvalues. We distinguish 3 cases depending on the number of simple eigenvalues of \dot{G} . First, if all of them are simple then all vertex-deleted subgraphs have fewer than 3 eigenvalues; this case is trivial. If \dot{G} has 2 simple eigenvalues, then \dot{G} is switching equivalent to a complete bipartite graph, by Lemma 5. If so, then every vertex-deleted subgraph is also switching equivalent to a complete bipartite graph; such a subgraph has 3 eigenvalues unless \dot{G} is the star $\dot{K}_{1,n-1}$ and the degree of the deleted vertex is $n - 1$. The remaining case is more complicated and it is considered in the following two theorems.

Theorem 10. *Let \dot{G} be a connected signed graph with spectrum $[\rho, \mu^m, \lambda^l]$, with $m, l \geq 2$ and $\mu > \lambda$. Let $\dot{H} = \dot{G} - v$ and let \mathbf{r} be the characteristic $(0, 1, -1)$ -vector that determines the neighbourhood of v in \dot{H} . If \dot{H} has 3 eigenvalues, then \mathbf{r} is an eigenvector associated with an eigenvalue of \dot{H} distinct from μ and λ , and:*

- (i) *for $\rho > \mu$, the spectrum of \dot{H} is $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ with $\|\mathbf{r}\|^2 = -\rho\lambda$;*
- (ii) *for $\rho < \lambda$, the spectrum of \dot{H} is $[\mu^{m-1}, \rho + \mu, \lambda^l]$ with $\|\mathbf{r}\|^2 = -\rho\mu$;*
- (iii) *for $\rho \in (\lambda, \mu)$, the spectrum of \dot{H} is $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$ with $\rho = \mu + \lambda$, $\|\mathbf{r}\|^2 = -\mu\lambda$ or $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ with $\|\mathbf{r}\|^2 = -\rho\lambda$, or $[\mu^{m-1}, \rho + \mu, \lambda^l]$ with $\|\mathbf{r}\|^2 = -\rho\mu$.*

Conversely, if \mathbf{r} is an eigenvector of \dot{H} associated with an eigenvalue distinct from μ and λ , then \dot{H} has 3 eigenvalues when either ρ is an eigenvalue of multiplicity 2 in \dot{H} or \dot{H} does not have ρ as an eigenvalue.

Proof. Computing $\text{tr}(A_{\dot{G}})$ and $\text{tr}(A_{\dot{G}}^2)$, we obtain

$$(7) \quad \rho + m\mu + l\lambda = 0,$$

$$(8) \quad \rho^2 + m\mu^2 + l\lambda^2 = 2e,$$

where e denotes the number of edges of \dot{G} . Let g be the number of edges in \dot{G} but not in \dot{H} , so that $g = \|\mathbf{r}\|^2$.

Suppose first that $\rho > \mu$. Observe that $\lambda < 0$, since \dot{G} must have at least one negative eigenvalue (see Proposition 1), and then we also have $\mu \geq 0$. By eigenvalue interlacing, the eigenvalues of \dot{H} are $\nu, \mu^{m-1}, \theta, \lambda^{l-1}$, where $\rho \geq \nu \geq \mu \geq \theta \geq \lambda$. Now, since \dot{H} has 3 eigenvalues we have one of the following situations:

- (a) $\nu = \rho$ and $\theta \in \{\mu, \lambda\}$,
- (b) $\nu = \mu$ and $\theta \notin \{\mu, \lambda\}$,
- (c) $\nu \notin \{\rho, \mu\}$ and $\theta = \mu$,
- (d) $\nu \notin \{\rho, \mu\}$ and $\theta = \lambda$.

For (a), when $\theta = \mu$ we have $\rho + m\mu + (l - 1)\lambda = 0$, which together with (7) leads to the contradiction $\lambda = 0$. When $\theta = \lambda$ in a similar way we get $\mu = 0$, but then \dot{H} and \dot{G} have the same number of edges (by (8)), a contradiction since \dot{G} is connected.

For (b), we use the equalities (7) and (8) along with $\text{tr}(A_{\dot{H}}) = 0$ and $\text{tr}(A_{\dot{H}}^2) = 2(e - g)$ to obtain $\theta = \rho + \lambda$ and $g = -\rho\lambda$. To complete the proof of (i) it remains to prove that \mathbf{r} is an eigenvector of \dot{H} associated with $\rho + \lambda$.

Let Q_ξ denote the matrix of the orthogonal projection of \mathbb{R}^{n-1} onto the eigenspace of an eigenvalue ξ of \dot{H} . By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left(x - \frac{\|Q_\mu \mathbf{r}\|^2}{x - \mu} - \frac{\|Q_\theta \mathbf{r}\|^2}{x - \theta} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x - \lambda} \right).$$

Since the multiplicity of μ and λ in \dot{G} is not less than the multiplicity of the same eigenvalue in \dot{H} , we have $Q_\mu \mathbf{r} = Q_\lambda \mathbf{r} = \mathbf{0}$, which means that \mathbf{r} is orthogonal to the eigenspaces of μ and λ , equivalently \mathbf{r} belongs to the eigenspace of $\theta = \rho + \lambda$.

For (c), as in the previous case, we obtain $\nu = \rho + \lambda$ and $g = -\rho\lambda$, along with the conclusion that \mathbf{r} belongs to the eigenspace of ν .

For (d), we find that $\nu = \rho + \mu$, which is impossible as $\mu \geq 0$ and $\nu \neq \rho$.

This completes the proof of (i), while (ii) follows analogously.

Now suppose that $\rho \in (\lambda, \mu)$. Here $\mu > 0, \lambda < 0$ and the possible eigenvalues of \dot{H} are $\mu^{m-1}, \nu, \theta, \lambda^{l-1}$. The cases that arise are considered in the same way as before, and yield the results summarized in (iii).

Assume now that \mathbf{r} belongs to the eigenspace of an eigenvalue of \dot{H} distinct from μ and λ . If ρ is an eigenvalue of \dot{H} with multiplicity 2, then (by eigenvalue interlacing) \dot{H} has 3 eigenvalues, and so it remains to consider the case in which ρ does not belong to the spectrum of \dot{H} . In this case \dot{H} has at most 4 eigenvalues. If λ, μ are the only eigenvalues of \dot{H} then $\rho = 0$ and we obtain the contradiction $g = 0$. Now suppose that \dot{H} has 4 eigenvalues, μ, λ, ν and θ . By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left(x - \frac{\|Q_\mu \mathbf{r}\|^2}{x - \mu} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x - \lambda} - \frac{\|Q_\nu \mathbf{r}\|^2}{x - \nu} - \frac{\|Q_\theta \mathbf{r}\|^2}{x - \theta} \right).$$

If \mathbf{r} belongs to the eigenspace of (say) ν , then we obtain

$$(9) \quad (x - \nu)(x - \rho)(x - \mu)^m(x - \lambda)^l = (x(x - \nu) - \|Q_\nu \mathbf{r}\|^2)P_{\dot{H}}(x).$$

Observe that the multiplicities of μ and λ in \dot{H} are $m - 1$ and $l - 1$, respectively, which implies $(x - \mu)(x - \lambda) = (x(x - \nu) - \|Q_\nu \mathbf{r}\|^2)$; but then since $(x - \rho)$ must appear on the right hand side of (9), we conclude that ρ is an eigenvalue of \dot{H} . This contradiction completes the proof. ■

We note a consequence of Theorem 10 in the case that \dot{G} is connected and switching equivalent to its underlying graph. Then ρ is the largest eigenvalue of \dot{G} , and so ρ is not an eigenvalue of \dot{H} . Hence, if \mathbf{r} is an eigenvector of \dot{H} associated with an eigenvalue other than μ or λ then \dot{H} has 3 eigenvalues. Plenty of examples can be found among unsigned graphs; for instance \dot{H} can be the Petersen graph, with \dot{G} the cone over \dot{H} . We further consider cones by setting $\dot{G} \cong K_1 \nabla \dot{H}$ in Theorem 10.

Corollary 11. *Suppose that $K_1 \nabla \dot{H}$ has spectrum $[\rho, \mu^m, \lambda^l]$, where $m, l \geq 2$ and $\mu > \lambda$. If \dot{H} is connected with 3 eigenvalues, then \dot{H} is net-regular and either:*

- (i) \dot{H} has spectrum $[\rho + \lambda, \mu^m, \lambda^{l-1}]$, with $\rho > \rho + \lambda > \mu > \lambda$ or
- (ii) \dot{H} has spectrum $[\mu^{m-1}, \lambda^l, \rho + \mu]$, with $\rho < \rho + \mu < \lambda < \mu$.

Proof. By setting $\mathbf{r} = \mathbf{j}$ in Theorem 10 we see that \mathbf{j} is an eigenvector of \dot{H} , and so \dot{H} is net-regular. Let $A_{\dot{H}}\mathbf{j} = \nu\mathbf{j}$, $\dot{G} \cong K_1\nabla\dot{H}$ and $n = 1 + l + m$. From Theorem 10 we see that $\nu \notin \{\mu, \lambda\}$ and there are five possible scenarios:

- (a) $\rho > \mu$, $\nu = \rho + \lambda$, $n - 1 = -\rho\lambda$ and \dot{H} has spectrum $[\mu^m, \rho + \lambda, \lambda^{l-1}]$;
- (b) $\rho < \lambda$, $\nu = \rho + \mu$, $n - 1 = -\rho\mu$ and \dot{H} has spectrum $[\mu^{m-1}, \rho + \mu, \lambda^l]$;
- (c) $\lambda < \rho < \mu$, $\nu = \rho + \lambda$, $n - 1 = -\rho\lambda$ and \dot{H} has spectrum $[\mu^m, \rho + \lambda, \lambda^{l-1}]$;
- (d) $\lambda < \rho < \mu$, $\nu = \rho + \mu$, $n - 1 = -\rho\mu$ and \dot{H} has spectrum $[\mu^{m-1}, \rho + \mu, \lambda^l]$;
- (e) $\lambda < \rho < \mu$, $\nu = \rho$, $n - 1 = -\lambda\mu$, $\rho = \mu + \lambda$ and \dot{H} has spectrum $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$.

Note first that if λ has multiplicity $l - 1$ in \dot{H} then $l > 2$, for otherwise \dot{H} has 2 simple eigenvalues and, by Lemma 5, is switching equivalent to a complete bipartite graph, say $K_{r,s}$. Then $\mu = 0$ and $\{\nu, \lambda\} = \{-\sqrt{rs}, \sqrt{rs}\}$. Moreover, $r + s = n - 1 = -\rho\lambda = (\lambda - \nu)\lambda = 2rs$, whence $\dot{H} \cong K_2$, a contradiction. Similarly, if μ has multiplicity $m - 1$ in \dot{H} , then $m > 2$. Now we may apply [14, Theorem 5.5] to \dot{H} , because \dot{H} is net-regular, ν is a simple eigenvalue of \dot{H} and each of λ, μ has multiplicity ≥ 2 in \dot{H} .

For (a), by [14, Theorem 5.5] either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n - 1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have $\nu > \mu + \lambda$ and so $\lambda < \nu - \mu < \nu$, a contradiction. In the latter case, we have part (i) of this corollary.

For (b), we again get either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n - 1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have part (ii) of this corollary. In the latter case, μ is the largest eigenvalue of \dot{G} because $\rho < \lambda$. Hence $\nu \leq \mu$, a contradiction.

For (c), we have $\lambda < 0$ and $\mu > 0$, because λ and μ are respectively the least and largest eigenvalues of \dot{G} . By [14, Theorem 5.5], either $\mu(\mu - \lambda) = n - 1$ or $\lambda(\lambda - \mu) = n - 1$. Since $\mu > \rho$, we have $-\lambda\mu > -\lambda\rho$, and so in the former case $\mu(\mu - \lambda) < -\lambda\mu$. Since $\mu > 0$, we have $\mu - \lambda < -\lambda$ and the contradiction $\mu < 0$. In the latter case, we have $\lambda(\lambda - \mu) = n - 1 = -\rho\lambda$, whence $\rho = \mu - \lambda > \mu$, a contradiction.

For (d), we have $\lambda < 0$, $\mu > 0$ and either $\mu(\mu - \lambda) = n - 1$ or $\lambda(\lambda - \mu) = n - 1$. In the former case, $\mu(\mu - \lambda) = n - 1 = -\rho\mu$, whence $\rho = \lambda - \mu < \lambda$, a contradiction. In the latter case, $\rho > \lambda$ and so $\lambda(\lambda - \mu) = -\rho\mu < -\lambda\mu$, whence $\lambda - \mu > -\mu$ and the contradiction $\lambda > 0$.

For (e), we have $\nu = \lambda + \mu$, and so by [14, Theorem 5.5], \dot{G} has just 2 eigenvalues, a contradiction. ■

It remains to consider signed graphs with 3 eigenvalues such that none of them is simple.

Theorem 12. *Let \dot{G} be a connected signed graph with spectrum $[\rho^r, \mu^m, \lambda^l]$, with $r, m, l \geq 2$. Let $\dot{H} = \dot{G} - v$ and let \mathbf{r} denote the characteristic $(0, 1, -1)$ -vector that determines the neighbourhood of v in \dot{H} .*

If \dot{H} has 3 eigenvalues then, to within a permutation of the eigenvalues, its spectrum is $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$; moreover, $\rho = \mu + \lambda$, $\|\mathbf{r}\|^2 = -\mu\lambda$, and \mathbf{r} belongs to the eigenspace of ρ in \dot{H} .

Conversely, if \mathbf{r} belongs to the eigenspace of an eigenvalue of \dot{H} , then \dot{H} has at most 4 eigenvalues, with 3 eigenvalues precisely when the eigenvalue associated with \mathbf{r} also belongs to the spectrum of \dot{G} .

Proof. Assume that \dot{H} has 3 eigenvalues and suppose first that the multiplicities of two of them (say μ and λ) are transferred from \dot{G} . Since $\text{tr}(A_{\dot{G}}) = \text{tr}(A_{\dot{H}})$ we have $\rho = 0$, and since $\text{tr}(A_{\dot{G}}^2) = \text{tr}(A_{\dot{H}}^2)$ we see that \dot{G} and \dot{H} have the same number of edges, a contradiction. By eigenvalue interlacing, the remaining case is the one in which every eigenvalue changes its multiplicity: one increases and two decrease. If the multiplicity of ρ increases, then as before we have $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that ρ is afforded by \mathbf{r} in \dot{H} and that $\|\mathbf{r}\|^2 = -\mu\lambda$.

Conversely, suppose that \mathbf{r} belongs to the eigenspace of the eigenvalue ξ of \dot{H} . By Proposition 3, we have

$$(10) \quad (x - \xi)P_{\dot{G}}(x) = P_{\dot{H}}(x)(x(x - \xi) - \|Q_{\xi}\mathbf{r}\|^2).$$

If \dot{H} has 5 eigenvalues (more than this is impossible), those transferred from \dot{G} along with (say) ν and θ , then at least one of the factors $(x - \nu)$ and $(x - \theta)$ occurs only on the right hand side of (10), which is impossible. Therefore, the number of eigenvalues of \dot{H} is 3 or 4. If their number is 3, then we see immediately that ξ is an eigenvalue of \dot{G} . Conversely, if ξ is an eigenvalue of \dot{G} , and ν is the fourth eigenvalue of \dot{H} (the one distinct from ρ, μ, λ), then as before $(x - \nu)$ occurs only on the right hand side of (10), which is impossible, and we are done. ■

Example 13. To obtain an example for Theorem 12 it is convenient to start from \dot{H} as a signed graph with spectrum $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$, and set $\mathbf{r} = \mathbf{j}$. In this case, \dot{H} is net-regular, and \dot{G} is a cone over \dot{H} . Moreover, $\rho = \mu + \lambda$ and \dot{H} has $-\mu\lambda$ vertices. By inspecting some known net-regular signed graphs with 3 eigenvalues, we arrive at a signed graph which can be found in [1] and satisfies all the numerical constraints. This is the signed graph obtained by reversing the sign of every edge belonging to a fixed Hamiltonian cycle of the Paley graph with 9 vertices. Its spectrum is $[3^2, 0^5, (-3)^2]$ (with $\rho = 0$). The corresponding cone has the spectrum $[3^3, 0^4, (-3)^3]$.

Motivated by the previous example, in which \dot{G} is a cone over \dot{H} , we give a closer description of both signed graphs in this particular case. The first statement of the next theorem is a general one.

Theorem 14. *The following statements hold.*

- (i) *Let \dot{G} be a signed graph with ρ as an eigenvalue of multiplicity $r \geq 2$, let $\dot{H} = G - v$ and let \mathbf{r} be the characteristic vector that determines the neighbourhood of v in \dot{H} . If the eigenspace of ρ in \dot{H} has orthogonal basis $\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ and v is vertex 1 in \dot{G} , then $(0, \mathbf{x}_1^\top)^\top, (0, \mathbf{x}_2^\top)^\top, \dots, (0, \mathbf{x}_k^\top)^\top$ are linearly independent eigenvectors associated with ρ in \dot{G} .*
- (ii) *In particular, if $\dot{G} \cong K_1 \nabla \dot{H}$ with spectrum $[\rho^r, \mu^m, \lambda^l]$ ($r, m, l \geq 2$), if \mathbf{j} belongs to the eigenspace of ρ in \dot{H} and if \dot{H} has spectrum $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$, then ρ is the unique main eigenvalue of \dot{H} and the unique non-main eigenvalue of \dot{G} .*

Proof. We have

$$A_{\dot{G}} \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix} = \begin{pmatrix} \mathbf{r}^\top \mathbf{x}_i \\ \rho \mathbf{x}_i \end{pmatrix} = \rho \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix},$$

for $1 \leq i \leq k$. Linear independence follows directly, and we have (i).

For (ii), first note that ρ is the unique main eigenvalue of \dot{H} because \mathbf{j} is an eigenvector of ρ in \dot{H} . Taking $k = r$ in (i) we obtain a basis for the eigenspace of ρ in \dot{G} consisting of vectors orthogonal to \mathbf{j} . Thus ρ is non-main in \dot{G} . If \dot{G} has another non-main eigenvalue, then the remaining one is main, but this means that the cone \dot{G} is net-regular and hence the complete graph with just 2 eigenvalues, a contradiction. ■

5. CONSTRUCTIONS OF SIGNED GRAPHS WITH AT MOST 3 EIGENVALUES

Here we give some examples of signed graphs with 3 eigenvalues, along with applications of some results from Section 3. The first construction is based on weighing matrices, while the second one is based on symmetric 3-class association schemes.

5.1. Weighing matrices

Let $W = W(n, \alpha)$ be a weighing matrix of order n with weight α , i.e., an $n \times n$ $(0, 1, -1)$ -matrix such that $W^\top W = \alpha I$. For $1 \leq m \leq n$, we call the submatrix of W indexed by rows $1, 2, \dots, m$ and columns $1, 2, \dots, n$ a *partial weighing matrix* with *weighing extension* W .

Theorem 15. *Let W'_1 and W'_2 be two partial weighing matrices of size $m \times n$ with weighing extensions W_1, W_2 of weight α . The following block matrix has spectrum $[-\sqrt{\alpha^{n+m}}, \sqrt{\alpha^{n-m}}, 2\sqrt{\alpha^m}]$:*

$$(11) \quad \mathcal{A}_m(W_1, W_2) = \begin{pmatrix} O_m & W_1' & W_2' \\ W_1'^T & O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ W_2'^T & \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix}.$$

Proof. Let B be the matrix $(W_1' | W_2')$, and let

$$C = \begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix}.$$

By a straightforward computation we arrive at the following equality (see, for example, [10]):

$$(2\sqrt{\alpha}I_{2n} - C)^{-1} = \begin{pmatrix} \frac{2}{3\sqrt{\alpha}}I_n & \frac{1}{3\alpha\sqrt{\alpha}}W_1^T W_2 \\ \frac{1}{3\alpha\sqrt{\alpha}}W_2^T W_1 & \frac{2}{3\sqrt{\alpha}}I_n \end{pmatrix}.$$

Therefore $B(2\sqrt{\alpha}I_{2n} - C)^{-1}B^T = 2\sqrt{\alpha}I_m$, and it follows that the vectors

$$\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B^T \mathbf{x} \end{pmatrix},$$

for $\mathbf{x} \in \mathbb{R}^m$, lie in the eigenspace for the eigenvalue $2\sqrt{\alpha}$ of $\mathcal{A}_m(W_1, W_2)$. Thus $2\sqrt{\alpha}$ is an eigenvalue of $\mathcal{A}_m(W_1, W_2)$ with multiplicity at least m . A similar argument applied for $C = O_n$ shows that $-\sqrt{\alpha}$ is an eigenvalue with multiplicity at least $n + m$. In addition, by eigenvalue interlacing, $\sqrt{\alpha}$ is an eigenvalue of $\mathcal{A}_m(W_1, W_2)$ with multiplicity at least $n - m$ since the submatrix

$$\begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^T W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^T W_1 & O_n \end{pmatrix}$$

has the spectrum $[-\sqrt{\alpha}^n, \sqrt{\alpha}^n]$. ■

This theorem enables us to construct an infinite family of signed graphs with spectrum $[-\sqrt{\alpha}^{n+m}, \sqrt{\alpha}^{n-m}, 2\sqrt{\alpha}^m]$ for some appropriate α . We remark that the matrix (11) does not always correspond to a signed graph. For a signed graph, the inner product of any row of W_1 and any row of W_2 has to be 0 or $\pm\sqrt{\alpha}$, and in [11] one can find a method for constructing a family of weighing matrices of weight 4 with this property. The method can be used to construct signed graphs with spectrum $[-2^{n+m}, 2^{n-m}, 4^m]$, as in the following example.

Example 16. Let W_1 and W_2 be as follows:

$$W_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

Considering the first two rows of W_1 and W_2 as the matrices W'_1 and W'_2 , we see from Theorem 15 that the signed graph \dot{G} with adjacency matrix $\mathcal{A}_2(W_1, W_2)$ has spectrum $[-2^8, 2^4, 4^2]$. Since \dot{G} has no vertices of degree 4, Theorem 4.3 shows that if a vertex-deleted subgraph of \dot{G} has 3 eigenvalues, then it is obtained by deleting a vertex of degree 8. There are exactly two such vertices, labelled by 1 and 2 in (11), and the deletion of either leads to a subgraph with spectrum $[-2^7, 2^5, 4]$. This is because, in the notation of Theorem 4.3, \mathbf{r} is an eigenvector of the vertex-deleted subgraph corresponding to the eigenvalue 2.

5.2. Symmetric 3-class association schemes

A symmetric 3-class association scheme \mathcal{R} consists of a set X and a partition of $X \times X$ into 4 non-empty binary relations R_0, R_1, R_2, R_3 satisfying the following constraints:

- $R_0 = \{(x, x) \mid x \in X\}$;
- If $(x, y) \in R_i$, then $(y, x) \in R_i$ and if $(x, y) \in R_k$, then the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k depending on i, j, k , but not on a particular choice of x, y .

For $0 \leq i \leq 3$, we define the $(0, 1)$ -matrix A_i with rows and columns indexed by the elements of X , and (x, y) -entry 1 if and only if $(x, y) \in R_i$. It follows that $A_0 = I$ and $A_i A_j = \sum_{k=0}^3 p_{ij}^k A_k$. For $i \in \{1, 2, 3\}$ let G_i be the graph with adjacency matrix A_i , and for distinct $i, j \in \{1, 2, 3\}$ let $\dot{G}_{i,j}$ be the signed graph with adjacency matrix $A_i - A_j$. The matrices A_i span a 4-dimensional commutative \mathbb{R} -algebra (called the Bose-Mesner algebra, cf. [3, Chapter 17]). It follows that the signed graphs $\dot{G}_{i,j}$ have at most 4 eigenvalues.

Theorem 17. *Let $\dot{G}_{i,j}$ be a signed graph arising from a 3-class association scheme. Then $\dot{G}_{i,j}$ is strongly regular with parameters*

$$r = p_{ii}^0 + p_{jj}^0, \quad a = p_{ii}^i + p_{jj}^i - 2p_{ij}^i, \quad b = -2p_{ij}^j + p_{ii}^j + p_{jj}^j, \quad c = p_{ii}^k + p_{jj}^k - 2p_{ij}^k$$

where $\{i, j, k\} = \{1, 2, 3\}$. Moreover, $\dot{G}_{i,j}$ has 3 eigenvalues if and only if $a + b = 2c \neq 0$, and 2 eigenvalues if and only if $a + b = 2c = 0$.

Proof. The first assertion follows from [12, Theorem 2.2], and the second from Lemma 7. Lastly, \dot{G}_{ij} is not a complete graph since $A_k \neq 0$, and so the third assertion follows from [18, Theorem 4.2]. ■

We note that b is replaced by $-b$ in [12, Definition 1.4]. Now we are ready to provide some examples of signed graphs with at most 3 distinct eigenvalues.

Example 18. The 3-class *Johnson scheme* $J(n, 3)$ ($n \geq 6$), also known as the tetrahedral scheme, is defined on the 3-subsets of an n -set, with two subsets in the relation R_i if they intersect in $3 - i$ elements. The following scheme provides the numbers p_{ij}^k relevant to $\dot{G}_{1,3}$; they are obtained by a simple computation.

$$\begin{array}{cccccccccc}
 p_{11}^1 & p_{33}^1 & p_{13}^1 & p_{11}^3 & p_{13}^3 & p_{33}^3 & p_{11}^2 & p_{33}^2 & p_{13}^2 \\
 n - 2 & \binom{n-4}{3} & 0 & 0 & 3(n - 6) & \binom{n-6}{3} & 4 & \binom{n-5}{3} & n - 5
 \end{array}$$

Therefore,

$$\begin{aligned}
 p_{11}^1 + p_{33}^1 - 2p_{13}^1 - 2p_{13}^3 + p_{11}^3 + p_{33}^3 &= n - 2 + \binom{n-4}{3} - 6(n - 6) + \binom{n-6}{3}, \\
 2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) &= 8 + 2\binom{n-5}{3} - 4n + 20.
 \end{aligned}$$

Both of the above expressions are equal to $\frac{1}{3}(n-2)(n-7)(n-9)$. By Theorem 17, $\dot{G}_{1,3}$ has 2 eigenvalues when $n = 7$ or $n = 9$, and 3 eigenvalues otherwise.

Alternatively we can find the spectrum of $\dot{G}_{i,j}$ by using the following information from [2, 9]:

$$\begin{aligned}
 \text{Spec}(A_1) &= \left[3(n - 3), (2n - 9)^{n-1}, (n - 7)\binom{n}{2}^{-n}, -3\binom{n}{3}^{-\binom{n}{2}} \right], \\
 \text{Spec}(A_3) &= \left[\binom{n-3}{3}, (-n^2 + 9n)/2 - 10^{n-1}, (n - 5)\binom{n}{2}^{-n}, -1\binom{n}{3}^{-\binom{n}{2}} \right].
 \end{aligned}$$

Here the eigenvalues are ordered by common eigenvectors and so the eigenvalues of $A_1 - A_3$ are ρ, μ^{n-1} and $\lambda\binom{n}{3}^{-n}$, where ρ, μ, λ are not necessarily distinct, and

$$\rho = 3(n - 3) - \binom{n-3}{3}, \quad \mu = \frac{1}{2}(n^2 - 5n + 2), \quad \lambda = -2.$$

Note that G_1 is regular of degree $r_1 = 3(n - 3)$, G_3 is regular of degree $r_2 = \binom{n-3}{3}$ and the graph underlying $\dot{G}_{1,3}$ is regular of degree $r_1 + r_2$.

We find that $\rho \geq \mu$ if and only if $n(n - 2)(n - 7) \leq 0$. When $n = 7$, $\dot{G}_{1,3}$ has spectrum $[8^7, -2^{28}]$ and by [14, Corollary 4.4] each vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues. Such a subgraph necessarily has spectrum $[6, 8^6, -2^{27}]$. When $n = 6$ the spectrum of $\dot{G}_{1,3}$ is $[8, 4^5, -2^{14}]$ and each vertex of $\dot{G}_{1,3}$ has degree 10. Since $-\rho\lambda \neq 10$, Theorem 10(i) shows that no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues.

Secondly, we find that $\rho \leq \lambda$ if and only if $(n - 1)(n - 2)(n - 9) \geq 0$. When $n = 9$, the spectrum of $\dot{G}_{1,3}$ is $[19^8, -2^{76}]$ and by [14, Corollary 4.4] each vertex-deleted subgraph has spectrum $[17, 19^7, -2^{75}]$. When $n > 9$, no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues for otherwise, in the notation of Theorem 10(ii), we have $\|\mathbf{r}\|^2 = -\rho\mu$; but $r_1 + r_2 = -\rho\mu$ if and only if $(n - 1)(n - 2)(n - 9) = 0$, equivalently $n = 9$.

The remaining case is $n = 8$, when Theorem 10(iii) shows in similar fashion that a vertex-deleted subgraph of $\dot{G}_{1,3}$ does not have 3 distinct eigenvalues. In summary, a vertex-deleted subgraph of the signed graph $\dot{G}_{1,3}$ derived from $J(n, 3)$ has 3 eigenvalues if and only if n is 7 or 9.

The definition of $J(n, 3)$ may be extended to $J(5, 3)$, but this scheme is degenerate in our context because then $A_3 = 0$. In fact, $J(5, 3)$ is a 2-class association scheme, with $G_1 \cong L(K_5)$ and $G_2 \cong \bar{G}_1$ (the Petersen graph). Here $\dot{G}_{1,2}$ has spectrum $[-3^5, 3^5]$, while any vertex-deleted subgraph has spectrum $[-3^4, 0, 3^4]$.

Example 19. The 3-class *Hamming scheme* $H(3, q)$ is defined on the triples of q symbols (words of length 3 over an alphabet with q letters), where two triples are in the relation R_i if they differ in i coordinates ($i = 0, 1, 2, 3$). In this case we have the following.

$$\begin{array}{cccccccccc}
 p_{11}^1 & p_{33}^1 & p_{13}^1 & p_{11}^3 & p_{33}^3 & p_{33}^3 & p_{11}^2 & p_{33}^2 & p_{13}^2 \\
 q - 2 & (q - 2)(q - 1)^2 & 0 & 0 & 3(q - 2) & (q - 2)^3 & 2 & (q - 2)^2(q - 1) & q - 1
 \end{array}$$

Therefore,

$$\begin{aligned}
 p_{11}^1 + p_{33}^1 - 2p_{13}^1 - 2p_{33}^3 + p_{11}^3 + p_{33}^3 &= q - 2 + (q - 2)(q - 1)^2 - 6(q - 2) + (q - 2)^3, \\
 2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) &= 4 + 2(q - 2)^2(q - 1) - 4q + 4.
 \end{aligned}$$

Note that both of the above expressions are equal to $2q(q - 2)(q - 3)$. By Theorem 17, the signed graph $\dot{G}_{1,3}$ has at most 3 distinct eigenvalues. Moreover, $c = 0$ if and only if q is 2 or 3, and these are the cases in which $\dot{G}_{1,3}$ has only 2 distinct eigenvalues. Thus, by [14, Corollary 4.4] any vertex deleted subgraph of $\dot{G}_{1,3}$ has 3 distinct eigenvalues when $q \in \{2, 3\}$. On the other hand, by the proof of [8, Theorem 4.2] the eigenvalues of $\dot{G}_{1,3}$ other than its net-degree are the roots of the quadratic

$$x^2 + \frac{b - a}{2}x + \frac{a + b}{2} - r.$$

Now, by Theorem 17, we conclude that the eigenvalues of $\dot{G}_{1,3}$ are

$$\rho = 3(q - 1) - (q - 1)^3, \quad \mu = q^2 - 2, \quad \lambda = -2.$$

For $q > 3$, we find $\rho < \lambda$, and so by Theorem 10(ii), if a vertex-deleted subgraph of $\dot{G}_{1,3}$ has only 3 distinct eigenvalues then $\|\mathbf{r}\|^2 = -\rho\mu$. This equality holds if

and only if

$$q(q-1)^2(q+2)(q-3) = 0.$$

Accordingly, no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues when $q > 3$.

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REFERENCES

- [1] M. Anđelić, T. Koledin and Z. Stanić, *On regular signed graphs with three eigenvalues*, Discuss. Math. Graph Theory **40** (2020) 405–416.
<https://doi.org/10.7151/dmgt.2279>
- [2] J. Araujo and T. Bratten, *On the spectrum of the Johnson graphs $J(n, k, r)$* , Congr. “Dr. Antonio A.R. Monteiro” **XIII** (2015) 57–62.
- [3] P.J. Cameron and J.H. van Lint, *Designs, Graphs, Codes and their Links* (Cambridge University Press, Cambridge, 1991).
<https://doi.org/10.1017/CBO9780511623714>
- [4] X.-M. Cheng, A.L. Gavriluyk, G.R.W. Greaves and J.H. Koolen, *Biregular graphs with three eigenvalues*, European J. Combin. **56** (2016) 57–80.
<https://doi.org/10.1016/j.ejc.2016.03.004>
- [5] X.-M. Cheng, G.R.W. Greaves and J.H. Koolen, *Graphs with three eigenvalues and second largest eigenvalue at most 1*, J. Combin. Theory Ser. B **129** (2018) 55–78.
<https://doi.org/10.1016/j.jctb.2017.09.004>
- [6] D. Cvetković, P. Rowlinson and S. Simić, *An Introduction to the Theory of Graph Spectra* (Cambridge University Press, Cambridge, 2010).
- [7] E.R. van Dam, *Nonregular graphs with three eigenvalues*, J. Combin. Theory Ser. B **73** (1998) 101–118.
<https://doi.org/10.1006/jctb.1998.1815>
- [8] T. Koledin and Z. Stanić, *On a class of strongly regular signed graphs*, Publ. Math. Debrecen **97** (2020) 353–365.
<https://doi.org/10.5486/PMD.2020.8760>
- [9] L. Lovász, *On the Shannon capacity of a graph*, IEEE Trans. Inform. Theory **25** (1979) 1–7.
<https://doi.org/10.1109/TIT.1979.1055985>
- [10] T.-T. Lu and S.-H. Shiou, *Inverses of 2×2 block matrices*, Comput. Math. Appl. **43** (2002) 119–129.
[https://doi.org/10.1016/S0898-1221\(01\)00278-4](https://doi.org/10.1016/S0898-1221(01)00278-4)
- [11] F. Ramezani, *Some regular signed graphs with only two distinct eigenvalues*, Linear Multilinear Algebra (2020), in-press.
<https://doi.org/10.1080/03081087.2020.1736979>

- [12] F. Ramezani and Y. Bagheri, *Constructing strongly regular signed graphs*, Ars. Combin., in-press.
- [13] F. Ramezani, P. Rowlinson and Z. Stanić, *On eigenvalue multiplicity in signed graphs*, Discrete Math. **343** (2020) 111982.
<https://doi.org/10.1016/j.disc.2020.111982>
- [14] F. Ramezani, P. Rowlinson and Z. Stanić, *Signed graphs with at most three eigenvalues*, Czechoslovak Math. J., in-press.
- [15] P. Rowlinson, *The main eigenvalues of a graph: A survey*, Appl. Anal. Discrete Math. **1** (2007) 445–471.
<https://doi.org/10.2298/AADM0702445R>
- [16] P. Rowlinson, *On graphs with just three distinct eigenvalues*, Linear Algebra Appl. **507** (2016) 462–473.
<https://doi.org/10.1016/j.laa.2016.06.031>
- [17] P. Rowlinson, *More on graphs with just three distinct eigenvalues*, Appl. Anal. Discrete Math. **11** (2017) 74–80.
<https://doi.org/10.2298/AADM161111033R>
- [18] Z. Stanić, *On strongly regular signed graphs*, Discrete Appl. Math. **271** (2019) 184–190.
<https://doi.org/10.1016/j.dam.2019.06.017>
- [19] Z. Stanić, *Main eigenvalues of real symmetric matrices with application to signed graphs*, Czechoslovak Math. J. **70** (2020) 1091–1102.
<https://doi.org/10.21136/CMJ.2020.0147-19>
- [20] T. Zaslavsky, *Matrices in the theory of signed simple graphs*, in: Advances in Discrete Mathematics and Applications, Mysore 2008, B.D. Acharya, G.O.H. Katona, J. Nešetřil (Ed(s)), (Mysore, Ramanujan Math. Soc., 2010) 207–229.

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