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REGULAR GRAPHS WITH SMALL SECOND LARGEST EIGENVALUE

Tamara Koledin, Zoran Stanić

We consider regular graphs with small second largest eigenvalue (denoted by λ_2). In particular, we determine all triangle-free regular graphs with $\lambda_2 \leq \sqrt{2}$, all bipartite regular graphs with $\lambda_2 \leq \sqrt{3}$, and all bipartite regular graphs of degree 3 with $\lambda_2 \leq 2$.

1. INTRODUCTION

The characteristic polynomial and the eigenvalues of a simple graph G are defined as the characteristic polynomial and the eigenvalues of its adjacency matrix A $(= A(G))$. If G has n vertices, then its eigenvalues, in non-increasing order, are denoted by $\lambda_1\ (= \lambda_1(G)), \lambda_2\ (= \lambda_2(G)), \ldots, \lambda_n\ (= \lambda_n(G))$.

There is a number of results concerning graphs with small second largest eigenvalue. The graphs whose second largest eigenvalue is at most $\sqrt{2} - 1$ are determined [**13**]. It is proven in [**3**] that the set of minimal forbidden subgraphs for $\lambda_2 \leq \dfrac{\sqrt{5}-1}{2}$ is finite, and the structure of these subgraphs is described. In addition, there are many results in which the upper bound for λ_2 does not exceed 1 (see [**14**] or [**9**]), or 2. The graphs with $\lambda_2 \leq 2$ are called reflexive [**13**]. More details on graphs with small second largest eigenvalue including their applications can be found in [**2**], [**13**], or [**4**]. Here we just recall that the second largest eigenvalue plays an important role in determining the structure of regular graphs. In addition, regular graphs with small second largest eigenvalue often have high connectivity properties, and they are relevant to theoretical computer science, the designs of robust computer networks, the theory of error correcting codes, and to complexity theory [**8**].

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In our previous work we considered regular graphs with $\lambda_2 \leq 1$ (see [**14**] and [**9**]). We now increase this bound, and we completely determine all triangle-free regular graphs with $\lambda_2 \leq \sqrt{2}$, all bipartite regular graphs with $\lambda_2 \leq \sqrt{3}$, and all bipartite reflexive cubic (i.e. regular graphs of degree 3) graphs. We use both theoretical and computational methods; for the computer search we use the GENREG (a program for fast generation of regular graphs developed by M. Meringer [**11**]).

The paper is organized as follows. In Section 2 we fix the notation and mention some results from the literature in order to make the paper more self-contained. In the next three sections we consider regular graphs whose second largest eigenvalue does not exceed $\sqrt{2}$, $\sqrt{3}$, and 2, respectively. In Section 6 we give some additional data on obtained graphs.

2. PRELIMINARIES

A path, cycle and a complete graph on n vertices will be denoted by P_n, C_n and K_n, respectively. A complete bipartite graph with n_1 (resp. n_2) vertices in the first (resp. second) colour class will be denoted by K_{n_1,n_2}. The complement of G is denoted by $\overline{G}$. We will use '$\dot\cup$' to denote the disjoint union of two graphs, while '$\cup$' will stand for the union of two sets. The graph consisting of k disjoint copies of G will be denoted by kG. The set of vertices of G will be denoted by X_G $(= X)$. If $S \subset X$ then $G[S]$ denotes the induced subgraph determined by S. The degree of a regular graph G will be denoted by r_G $(= r)$, while the corresponding graph will be called r-regular. The set of neighbours of a vertex $v \in G$ will be denoted by $N(v)$, and then we denote $N[v] = N(v) \cup \{v\}$. For the remaining notation we refer the reader to [**2**].

The *bipartite complement* of connected bipartite graph G with two colour classes U and W is bipartite graph $\overline{\overline{G}}$ with the same color classes having the edge between U and W exactly where G does not. If G is bipartite r-regular graph on $2n$ vertices, and adjacency matrix

$$A(G) = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}, \tag{1}$$

then $\overline{\overline{G}}$ is bipartite $(n-r)$-regular graph, with adjacency matrix

$$A(\overline{\overline{G}}) = \begin{pmatrix} 0 & J-N \\ J-N^T & 0 \end{pmatrix},$$

where J is all-1 matrix. By [**15**, Theorem 4.1], the characteristic polynomials of G and $\overline{\overline{G}}$ satisfy

$$\frac{P_G(x)}{x^2-r^2} = \frac{P_{\overline{\overline{G}}}(x)}{x^2-(n-r)^2},$$

and so apart from the eigenvalues $\pm r$ of G and $\pm(n-r)$ of $\overline{\overline{G}}$, the spectra of G and $\overline{\overline{G}}$ are the same. Note that if G is disconnected then its bipartite complement is not uniquely determined, but even then the above formula remains unchanged.

Finally, we describe the *LCF notation* (see [**1**]) that will be used in Section 4, and Section 5. If G is a Hamiltonian cubic graph then its vertices can be arranged in a cycle C which accounts for two edges per arbitrary vertex v. The third edge vu can be described by the length l of the path vu in C with a sign plus if we turn clockwise, or minus if we turn counterclockwise along C. If the pattern of the LCF notation repeats, it is indicated by a superscript in the notation. If the second half of the numbers of the LCF notation is the reverse of the first half, but with all the signs changed, then it is replaced by a semicolon and a dash. Obviously, this notation is not unique.

Note that the similar notation can be used for other Hamiltonian r-regular graphs, but now for each vertex we have a set of $r-2$ numbers representing its adjacency in the way described above.

3. TRIANGLE-FREE REGULAR GRAPHS WITH $\lambda_2 \le \sqrt{2}$

In [**14**] and [**9**] we have described the procedure of determining regular graphs that satisfy the condition $\lambda_2 \le 1$, which enabled us to identify the resulting graphs in some particular cases. Here we rise up the corresponding bound to $\sqrt{2}$. Next, we include the additional condition within triangle-free graphs. Clearly, the degree of any regular graph satisfying $1 < \lambda_2 \le \sqrt{2}$ must be at least 2, and any such a graph is connected.

First we prove a general result.

Theorem 1. *Let G be a connected r-regular ($r \ge 3$) graph satisfying $\lambda_2 < \sqrt{r}$. Then* $\mathrm{diam}(G) \le 3$.

Proof. Let G be a connected r-regular graph with diameter at least 4, and consider two vertices u and v at distance 4. Then there are no edges between $N[u]$ and $N[v]$, and, by eigenvalue interlacing, $\lambda_2(G) \ge \lambda_2(G[N[u] \cup N[v]]) \ge \min(\lambda_1(G[N[u]]), \lambda_1(G[N[v]]) \ge \lambda_1(K_{1,r}) = \sqrt{r}$ (where the last inequality follows from [**2**, Theorem 0.7]), a contradiction. $\square$

According to the above theorem, the diameter of any r-regular graph ($r \ge 3$) with $\lambda_2 \le \sqrt{2}$ is at most 3.

Connected bipartite regular graphs with $\lambda_2 \le \sqrt{2}$ are determined in [**15**]. These graphs are $K_{r,r}$ ($r \ge 1$), $\overline{\overline{rK_2}}$ ($r \ge 3$), C_8, the Heawood graph and its bipartite complement (the Heawood graph has 14 vertices [**2**, p. 225]). Therefore, we consider next only non-bipartite triangle-free regular graphs satisfying the same condition. Since there are exactly two non-bipartite 2-regular graphs with $\lambda_2 \le \sqrt{2}$ (C_5 and C_7), in the following we can assume that $r \ge 3$.

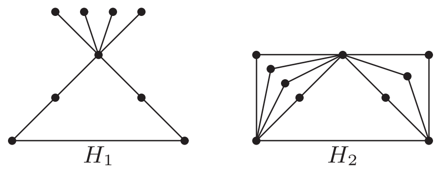

Figure 1. Two forbidden subgraphs for regular graphs with $\lambda_2 \le 2$.

First we establish an upper bound on r. We use the graphs H_1 and H_2 depicted in Fig. 1. Graph H_1 is forbidden subgraph for the property $\lambda_2 \leq \sqrt{2}$ in the family of regular graphs because the least eigenvalue of $\overline{H_1}$ is less than $-1-\sqrt{2}$ (and for every regular graph G, $\lambda_2(G) = -\lambda_n(\overline{G}) - 1$ holds), and H_2 is forbidden subgraph (in general) for the same property.

Lemma 1. *Let G be a non-bipartite triangle-free r-regular graph satisfying $\lambda_2 \leq \sqrt{2}$. Then $r \leq 5$.*

Proof. First, G cannot contain C_k ($k \geq 9$) as an induced subgraph since $\lambda_2(C_k) > \sqrt{2}$. Even more, if $r \geq 6$ then G does not contain C_7 as an induced subgraph (otherwise G contains $K_{1,6}\dot{\cup}K_2$ as an induced subgraph, but the least eigenvalue of its complement is less than $-1-\sqrt{2}$, implying $\lambda_2(G) > \sqrt{2}$). Since G is non-bipartite, it then must contain C_5 as an induced subgraph.

If $r \geq 10$, considering the neighbuorhood of a single vertex belonging to a C_5 (contained in G) we get that G must contain at least one of the graphs H_i ($i = 1, 2$) as an induced subgraph, and therefore $\lambda_2(G) > \sqrt{2}$.

Let further $6 \leq r \leq 9$ and let v be an arbitrary vertex of G. Denote by U the set of vertices of G which are at distance 2 from v (obviously U is non-empty), and for $u \in U$ define $d^*(u)$ to be the number of common neighbours of u and v ($d^*(u) \geq 1$). There are four cases to consider.

Case 1: $r = 9$. Suppose that U contains a vertex u with $d^*(u) = 3$. Then the induced subgraph $G[N(u)\triangle N(v)]$ (where $\triangle$ stands for symmetric difference) must be bipartite 3-regular (otherwise G contains either H_1 or H_2 as an induced subgraph) on 12 vertices. There are five connected such graphs (see [**2**, p. 300-306]), and one disconnected ($2K_{3,3}$), and for each of them $\lambda_2 > \sqrt{2}$.

Similarly, if U contains a vertex u with $d^*(u) = 4$ then $G[N(u)\triangle N(v)]$ must be bipartite 2-regular. It is clear that for such a graph either $\lambda_2 = 2$ (if it is disconnected), or $\lambda_2 = \frac{1+\sqrt{5}}{2} > \sqrt{2}$ (if it is connected).

Therefore, two vertices at distance 2 in G either have less than three, or more than four common neighbours. Suppose now that there is a vertex $u \in U$, with $1 \leq d^*(u) \leq 2$, and let $w \in N(u) \cap N(v)$. Now, each neighbour of u (not adjacent to v) has at least five neighbours among the neighbours of v (not adjacent to u), and vice versa (otherwise G contains H_1 as an induced subgraph). Since G cannot contain H_2 as an induced subgraph, each vertex in $N(u)\backslash N(v)$, as well as each vertex in $N(v)\backslash N(u)$ can be adjacent to at most one vertex in $N(w)\backslash\{u, v\}$, but then G contains H_1 as an induced subgraph. Thus, every two vertices at distance 2 in G must have more than four common neighbours, but then C_5 cannot be an induced subgraph of G, a contradiction.

Case 2: $r = 8$. Suppose that U contains a vertex u with $d^*(u) = 3$, and set $N(u)\backslash N(v) = \{w_1, \ldots, w_5\}$. For each $w_i \in N(u)\backslash N(v)$, $2 \leq d^*(w_i) \leq 3$ holds (otherwise G contains either H_1 or H_2 as an induced subgraph). Since there are five vertices in $N(u)\backslash N(v)$, there are either two of them satisfying $d^*(w_i) = 2$ and having at least one common neighbour in $N(v)\backslash N(u)$, or there are two of them satisfying $d^*(w_i) = 3$ and having at least two common neighbours in $N(v)\backslash N(u)$. By direct computation we get that each of the possible four subgraphs of G (induced by u, $N[v]$, and two vertices of $N(u)\backslash N(v)$ described above) has $\lambda_2 > \sqrt{2}$.

On the other hand, if U contains a vertex u with $1 \leq d^*(u) \leq 2$, let $w \in N(u) \cap N(v)$, and then the remainder of the proof is the same as the corresponding part in Case 1.

Case 3: $r = 7$. If $n \geq 28$ then, according to [**7**, Theorem 2.1.8], G does not contain $K_{2,3}$ as an induced subgraph, so for every $u \in U$, $d^*(u) \leq 2$ must hold. Let $u \in U$ be a vertex with $d^*(u) = 2$ (such a vertex must exist, otherwise G contains P_2 with six pendant edges attached to each of the two endvertices, and such a graph has $\lambda_2 > \sqrt{2}$). Now, the subgraph $G[N(u) \triangle N(v)]$ must be bipartite 2-regular (otherwise G contains either H_1 or $K_{2,3}$ as an induced subgraph), and for such a subgraph $\lambda_2 > \sqrt{2}$.

If $24 \leq n \leq 26$ then, according to [**7**, Theorem 2.1.8], G does not contain neither $K_{2,4}$, nor $K_{3,3}$ as an induced subgraph, so there is a vertex $u \in U$ with $d^*(u) = 3$. Of course, the three vertices in $N(u) \cap N(v)$ must have all their neighbours (distinct from u and v) in $X\backslash(N(u) \cup N(v))$. In addition, no vertex of G is adjacent to all three common neighbours of u and v.

If $n = 26$, since there are exactly 15 edges between $N(u) \cup N(v)$ and $X\backslash(N[u] \cup N[v])$, there must be at least two vertices adjacent to two common neighbours of u and v. Consider the partition of X in four parts: $A = N(u) \cap N(v)$, B is the set of the above mentioned two vertices (adjacent to two vertices of A), $C = \{u, v\}$, and $D = X\backslash(A \cup B \cup C)$. This partition induces the quotient matrix

$$Q_1 = \begin{pmatrix} 0 & 3 & 0 & 4 \\ 2 & 0 & \frac{4}{3} & \frac{11}{3} \\ 0 & 2 & 0 & 5 \\ \frac{8}{19} & \frac{11}{19} & \frac{10}{19} & \frac{104}{19} \end{pmatrix}.$$

According to [**7**, Theorem 1.2.3], we have $\lambda_2(G) > \lambda_2(Q_1) > \sqrt{2}$.

If $n = 24$ there are at least four vertices which do not belong to $N[u] \cup N[v]$, and which are adjacent to two common neighbours of u and v. Thus G contains $K_{2,4}$ as an induced subgraph, which is a contradiction.

If $n = 22$ then G cannot contain neither $K_{2,5}$, nor $K_{3,3}$ as an induced subgraph (cf. [**7**, Theorem 2.1.8]), so either $d^*(u) = 3$ for all $u \in U$, or there is a vertex $u \in U$ with $d^*(u) = 4$. In the first case, it is easy to check that $\mathrm{diam}(G) = 2$, so G must be a strongly regular graph with parameters $(22, 7, 0, 3)$, but such a graph does not exist (see, for example, [**2**, Theorem 7.3]). In the second case it can be easily verified that (under the assumption that G does not contain neither $K_{3,3}$, nor H_2 as an induced subgraph), a vertex $w \in X\backslash(N[u] \cup N[v])$ adjacent to exactly five vertices of $N(v)$ (or $N(u)$) must exist, but then the subgraph $G[u \cup w \cup N[v]]$ (or $G[v \cup w \cup N[u]]$) has $\lambda_2 > \sqrt{2}$.

If $n = 20$ then, according to [**7**, Theorem 2.1.8], G does not contain $K_{2,6}$, nor $K_{3,4}$ as an induced subgraph, so for every $u \in U$, $d^*(u) \leq 5$ must hold. Suppose first that there is a vertex $u \in U$ with the property $d^*(u) = 5$. Each vertex belonging to the set $N(u) \cap N(v)$ has all its neighbours (distinct from u and v) in $X \backslash (N(u) \cup N(v))$. Denote by $A = X \backslash (N[u] \cup N[v])$. Since $|A| = 9$ and each vertex in A can have at most 3 neighbours in $N(u) \cap N(v)$, counting the edges between the sets $N(u) \cap N(v)$ and A, we get that A either contains 7 vertices with three neighbours in $N(u) \cap N(v)$ and 2 vertices with two neighbours in $N(u) \cap N(v)$, or 8 vertices with three neighbours in $N(u) \cap N(v)$ and one vertex with one neighbour in $N(u) \cap N(v)$. In both cases the number of edges between A and $N(u) \triangle N(v)$ would be greater than 24, which is not possible.

Suppose now that there is a vertex $u \in U$ with the property $d^*(u) = 4$. Counting the edges between the sets $N(u) \cap N(v)$ and $A = X \backslash (N[u] \cup N[v])$ (note that now $|A| = 8$) we get that A must contain 4 vertices with three neighbours in $N(u) \cap N(v)$ and 4 vertices with two neighbours in $N(u) \cap N(v)$, which is not possible (otherwise, G would contain a vertex u' with $d^*(u') = 5$, and this case is already excluded).

The inequality $d^*(u) \leq 3$ cannot hold for each vertex $u \in U$, since there are 42 edges between the sets $N(v)$ and U.

There are exactly 8 triangle-free 7-regular graphs on 18 vertices. Computing their spectra, we get that none of them satisfies the property $\lambda_2 \leq \sqrt{2}$.

If $n \leq 16$ then G, since it is triangle-free, must be bipartite [**5**].

Case 4: $r = 6$. If $n \geq 22$ then, according to [**7**, Theorem 2.1.8], G does not contain $K_{2,3}$ as an induced subgraph, so either there are two vertices with just one common neighbour in which case the subgraph induced by their ten non-common neighbours must be bipartite 2-regular (otherwise G contains H_1 as an induced subgraph), but obviously, for such a subgraph $\lambda_2 > \sqrt{2}$ holds, or every two vertices at distance 2 in G have exactly two common neighbours. If so, then $\operatorname{diam}(G) = 2$, and thus G must be a strongly regular graph with parameters $(22, 6, 0, 2)$, but such a graph does not exist (see [**2**, Theorem 7.3] if necessary).

Suppose now that $n \in \{20, 21\}$. According to [**7**, Theorem 2.1.8], G does not contain $K_{2,4}$ nor $K_{3,3}$ as an induced subgraph. This implies that there is a vertex $u \in U$ with $d^*(u) = 3$. Now, all three vertices of $N(u) \cap N(v)$ must have their other neighbours in $S = X \backslash (N(u) \cup N(v))$. Since each vertex of S can be adjacent to at most two vertices in $N(u) \cap N(v)$, there must be at least two of them adjacent to exactly two vertices from the same set. Consider now the partition of X with $A = N(u) \cap N(v)$, B is the set of the above mentioned two vertices adjacent to two vertices of A, $C = \{u, v\}$, and $D = X \backslash (A \cup B \cup C)$. This partition induces the quotient matrix

$$Q_2 = \begin{pmatrix} 0 & 3 & 0 & 3 \\ 2 & 0 & \frac{4}{3} & \frac{8}{3} \\ 0 & 2 & 0 & 4 \\ \frac{6}{n-7} & \frac{8}{n-7} & \frac{8}{n-7} & \frac{6n-64}{n-7} \end{pmatrix}.$$

We get $\lambda_2(Q_2) > \sqrt{2}$ for $n \in \{20, 21\}$, which yields $\lambda_2(G) > \sqrt{2}$.

The cases $15 \leq n \leq 19$ can be considered in the similar way or by computer search. Using the latter option we get that among 4010 connected triangle-free 6-regular graphs obtained none satisfies the condition $\lambda_2 \leq \sqrt{2}$.

If $n \leq 14$ then G, since it is triangle-free, must be bipartite [**5**]. □

We have bounded degree of non-bipartite triangle-free regular graphs with $\lambda_2 \leq \sqrt{2}$ and the resulting cases are: $r = 3, 4, 5$. Now, in each of these cases we bound the number of vertices using the following inequality concerning triangle-free regular graphs [**10**]:

$$n \leq \frac{r^2(\lambda_2 + 2) - r\lambda_2(\lambda_2 + 1) - \lambda_2^2}{r - \lambda_2^2}.$$

We also use the known fact that if G is a simple triangle-free graph on n vertices and minimum degree d satisfying $d > \frac{2}{5}n$ then G must be bipartite (see [**5**]). This gives us the lower bound on the number of vertices in the non-bipartite case. Thus:

(i) if $r = 5$ then $14 \leq n \leq 22$,

(ii) if $r = 4$ then $10 \leq n \leq 19$,

(iii) if $r = 3$ then $8 \leq n \leq 18$.

Note that for $r = 5$ the cases $n = 20$ and $n = 22$ can be considered in a similar way as in Lemma 1, but here we verify the non-existence of the resulting graphs using computer search and the fact that the diameter of any putative graph is at most 3 (Theorem 1). For the remaining cases we use the brut force, i.e. we generate all possible connected non-bipartite triangle-free regular graphs using GENREG, and then we eliminate those with $\lambda_2 > \sqrt{2}$. The results are summarized in the following theorem.

Theorem 2. *There are exactly* 9 *non-bipartite triangle-free regular graphs whose second largest eigenvalue does not exceed* $\sqrt{2}$*. Apart from* C_5, C_7*, the Petersen graph, the Clebsch graph* (*it has* 16 *vertices* [**2**, p. 185]), *the remaining five graphs are depicted in Fig.* 2.

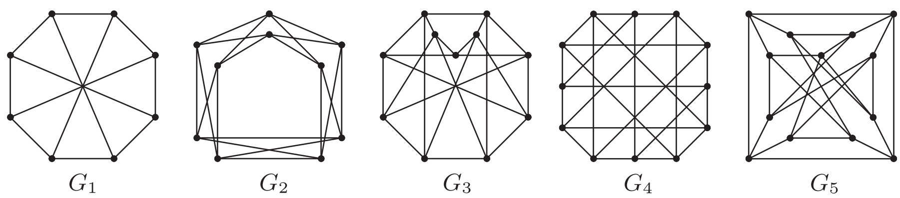

Figure 2. Five graphs from Theorem 2.

Collecting the results of [**15**] and Theorem 2, we get the following theorem.

Theorem 3. *There are two infinite families of connected triangle-free regular graphs with* $\lambda_2 \leq \sqrt{2}$ ($K_{r,r}$ $(r \geq 1)$*, and* $\overline{\overline{rK_2}}$ $(r \geq 3)$)*, and exactly* 12 *additional graphs*: C_8*, Heawood graph, its bipartite complement, and the graphs from Theorem* 2.

Spectra of graphs given in Theorem 3 are listed in Table 1 (here and below, the exponents stand for the multiplicity of the eigenvalue; graph and its bipartite complement are given in the same row).

graph	n	diam	spectrum
$K_{r,r}$ $(r \geq 1)$	$2r$	1 or 2	$r, 0^{2r-2}, -r$
$\overline{\overline{rK_2}}$ $(r \geq 3)$	$2r$	3	$r-1, 1^{r-1}, -1^{r-1}, -r+1$
C_8	8	4	$2, 1.41^2, 0^2, -1.41^2, -2$
Heawood($\overline{\overline{\text{Heawood}}}$)	14	3(3)	$3(4), 1.41^6, -1.41^6, -3(-4)$
C_5	5	2	$2, 0.62^2, -1.62^2$
C_7	7	3	$2, 1.25^2, -0.45^2, -1.80^2$
G_1	8	2	$3, 1^2, 0.41^2, -1, -2.41^2$
Petersen	10	2	$3, 1^5, -2^4$
G_2	10	2	$4, 1.23^2, 0^5, -3.23^2$
G_3	11	2	$4, 1.40^2, 0.55^2, 0.37^2, -1.09^2, -3.23^2$
G_4	12	2	$4, 1^6, 0, -2^2, -3^2$
G_5	13	2	$4, 1.38^4, 0.27^4, -2.66^4$
Clebsch	16	2	$5, 1^{10}, -3^5$

Table 1. Connected triangle-free regular graphs with $\lambda_2 \leq \sqrt{2}$.

4. BIPARTITE REGULAR GRAPHS WITH $\lambda_2 \leq \sqrt{3}$

We proceed to determine all bipartite regular graphs with the property $\lambda_2 \leq \sqrt{3}$. Let G be such a graph and, since any bipartite regular graph must have an even order, in this and in the next section we shall assume that G has $2n$ vertices. Since all bipartite regular graphs with $\lambda_2 \leq \sqrt{2}$ are already determined, we can restrict ourselves to bipartite regular graphs with $\sqrt{2} < \lambda_2 \leq \sqrt{3}$. Obviously, any such a graph is connected.

Lemma 2. *Let* G *be a bipartite* r*-regular graph satisfying* $\sqrt{2} < \lambda_2 \leq \sqrt{3}$*. Then* $r \leq 9$.

Proof. Suppose that $r \geq 10$, and denote two colour classes of G by U and W, respectively. If we fix one vertex $v \in U$ then we can partition W into two disjoint subsets: $A = N(v)$ of size r, and $B = W \backslash A$ of size $m = n - r$.

Let $m \geq 4$. Suppose that there is a vertex $u \in U$ adjacent to $\ell \geq 4$ vertices of B. Then A must contain at least ℓ vertices not adjacent to u. So G contains $2K_{1,4}$ as an induced subgraph, and therefore $\lambda_2(G) > \sqrt{3}$ holds. Consequently, every vertex in U is adjacent to at most 3 vertices of B. Counting the number of edges having one endvertex in B and those with one endvertex in U, we get

$$rm \leq 3(r + m - 1),$$

or $m \leq \dfrac{3r-3}{r-3}$, and (since $r \geq 10$) we have $m \leq \dfrac{27}{7} < 4$, which yields $|B| \leq 3$. Two cases arise: $m = 2$ and $m = 3$ (for $m \in \{0,1\}$, $\lambda_2(G) < \sqrt{2}$).

If $m = 2$ then $2n = 2(r+2)$. The bipartite complement of G is a bipartite 2-regular (possibly disconnected) graph. Since $r \geq 10$ we have $\lambda_2(G) = \lambda_2(\overline{\overline{G}}) > \sqrt{3}$.

If $m = 3$ then $2n = 2(r+3)$. A simple counting implies that there are at least $r-4$ vertices in U adjacent to all three vertices of B. Let $C \subseteq U$ be the set of $r-4$ such vertices. Consider the partition of the vertex set of G: v, A, C, $U \setminus (C \cup \{v\})$, B. The corresponding quotient matrix has the form

$$Q = \begin{pmatrix} 0 & r & 0 & 0 & 0 \\ 1 & 0 & \dfrac{r^2-7r+12}{r} & \dfrac{6(r-2)}{r} & 0 \\ 0 & r-3 & 0 & 0 & 3 \\ 0 & r-2 & 0 & 0 & 2 \\ 0 & 0 & r-4 & 4 & 0 \end{pmatrix}. \tag{2}$$

We get $\lambda_2(G) \geq \lambda_2(Q) = \sqrt{\dfrac{5r-12}{r}}$, which, together with the assumption $r \geq 10$, yields $\lambda_2(G) > \sqrt{3}$. A contradiction. □

Our next step is to bound the order of the desired graphs. We check that if $r = 2$ then $n \leq 6$.

Lemma 3. *Let G be a bipartite 3-regular graph on $2n$ vertices with $\sqrt{2} < \lambda_2 \leq \sqrt{3}$. Then $n \leq 9$.*

Proof. Suppose that $n \geq 10$. Then $\overline{\overline{G}}$ is $(n-3)$-regular, and considering the vertex partition of the vertex set of $\overline{\overline{G}}$ described in the previous lemma, we get the quotient matrix (2) with $r = n-3$, and thus $\lambda_2(G) = \lambda_2(\overline{\overline{G}}) \geq \sqrt{\dfrac{5n-27}{n-3}} > \sqrt{3}$, a contradiction. □

If $4 \leq r \leq 9$, using the inequality concerning r-regular graphs with diameter 3, $n \leq 2\dfrac{r^2-\lambda_2^2(G)}{r-\lambda_2^2(G)}$ [**10**, Theorem 3.2]), we get:

(i) if $r = 9$ then $n \leq 13$,
(ii) if $r = 8$ then $n \leq 12$,
(iii) if $r = 7$ then $n \leq 11$,
(iv) if $r = 6$ then $n \leq 11$,
(v) if $r = 5$ then $n \leq 11$,
(vi) if $r = 4$ then $n \leq 13$.

Considering the above cases, we get that if G and $\overline{\overline{G}}$ are bipartite regular graphs with $\sqrt{2} < \lambda_2(G) = \lambda_2(\overline{\overline{G}}) \leq \sqrt{3}$, and if $\overline{\overline{G}}$ satisfies one of the conditions (i)-(iv) then the degree of G is between 2 and 5. So, we can restrict our search to bipartite r-regular graphs with $\sqrt{2} < \lambda_2 \leq \sqrt{3}$, and $2 \leq r \leq 5$.

If $r = 2$, we easily get that the only resulting graphs are C_{10} and C_{12}.

If $r = 3$, we have $n \leq 9$. There are 209 connected bipartite regular graphs satisfying these conditions on r and n. Four of them satisfy $\sqrt{2} < \lambda_2 \leq \sqrt{3}$ ($\overline{\overline{C_{10}}}, I_1, I_2$, and I_3 of Table 2).

Let $r = 4$. It is sufficient to consider the cases $8 \leq n \leq 13$, since if $n \leq 7$ then any resulting graph is necessarily a bipartite complement of some already obtained solution with lower degree. If $n = 13$ the resulting graph is a unique bipartite 4-regular graph on 26 vertices and diameter 3 (namely, this graph must be the incidence graph of a symmetric balanced incomplete block design with parameters $(13, 4, 1)$, and there exists exactly one such design [**12**]) – I_4 of Table 2. There is a unique bipartite 4-regular graph on $2n = 24$ vertices and diameter 3 [**6**], but its second largest eigenvalue is greater than $\sqrt{3}$. Similarly, if $n = 11$ there are two corresponding graphs [**6**], but again $\lambda_2 > \sqrt{3}$ for both graphs.

If $8 \leq n \leq 10$, according to [**7**, Theorem 2.1.8], G does not contain $K_{2,3}$ as an induced subgraph. Also, according to Theorem 1, $\mathrm{diam}(G) \leq 3$, and since $\mathrm{diam}(G) > 2$ (the only connected bipartite regular graph with diameter 2 is the complete bipartite graph), we get $\mathrm{diam}(G) = 3$. This means that two vertices belonging to the same colour class of G must have either one or two common neighbours. Denote two colour classes of G by U and W, respectively, and let v be an arbitrary vertex belonging to U. Counting the edges between $N(v)$ and $U \setminus \{v\}$, we get the following facts:

1. if $n = 10$, the set $U \setminus \{v\}$ contains exactly 3 vertices with two common neighbours with v and 6 vertices with one common neighbour with v,

2. if $n = 9$, the set $U \setminus \{v\}$ contains exactly 4 vertices with two common neighbours with v and 4 vertices with one common neighbour with v,

3. if $n = 8$, the set $U \setminus \{v\}$ contains exactly 5 vertices with two common neighbours with v and 2 vertices with one common neighbour with v.

Let (1) be the adjacency matrix of G, and suppose first that $n = 10$ holds. Then $NN^T = 4I + 2M + (J - I - M)$, where M is the adjacency matrix of some 3-regular graph on 10 vertices. Since $\lambda_2(NN^T) = \lambda_2^2(G) \leq 3$, we get $\lambda_2(M) \leq 0$. The graph corresponding to M is obviously connected, and it must be complete multipartite (see [**2**, Theorem 6.7]), but of course, no such 3-regular graph on 10 vertices exists. Suppose now that $n = 9$. In the same way we get that then the graph corresponding to M must be a complete multipartite 4-regular graph on 9 vertices, and such a graph does not exist. Similarly, if $n = 8$, we get that a complete multipartite 5-regular graph on 8 vertices would exist, which is a contradiction.

Let $r = 5$. As in the previous case, it is sufficient to consider the cases $n = 11$, and $n = 10$. In the first case, any two vertices in the same colour class must have exactly two common neighbours (in any other case $\lambda_2(G) > \sqrt{3}$ holds). In other words, the resulting graph corresponds to a symmetric balanced incomplete block design with parameters $(11, 5, 2)$, and there is exactly one such design [**12**]. It is the graph I_5 of Table 2. If $n = 10$, then in the same way as in the cases $r = 4$, $8 \leq n \leq 10$, we can deduce that then a connected complete multipartite 2-regular graph on 10 vertices would exist, which is not possible.

Summarizing the above results we get the following theorem.

Theorem 4. *There are exactly* 13 *bipartite regular graphs satisfying* $\sqrt{2} < \lambda_2 \leq \sqrt{3}$. *They are listed in Table* 2 (*we use the LCF notation to represent the graphs* I_i $(i = 1, \dots, 5)$).

graph	$2n$	diam	LCF notation of I_i $(i = 1, \dots, 5)$	spectrum (non-negative part)
$C_{10}(\overline{\overline{C_{10}}})$	10	5(3)		$2(3), 1.62^2, 0.62^2$
$C_{12}(\overline{\overline{C_{12}}})$	12	6(3)		$2(4), 1.73^2, 1^2, 0^2$
I_1	12	3	$[5, -5]^6$	$3, 1.73^2, 1^3$
$I_2(\overline{\overline{I_2}})$	16	4(3)	$[5, -5]^8$	$3(5), 1.73^4, 1^3$
$I_3(\overline{\overline{I_3}})$	18	4(3)	$[5, 7, -7; -]^3$	$3(6), 1.73^6, 0^4$
$I_4(\overline{\overline{I_4}})$	26	3(3)	$[\{-7, -11\}, \{7, 11\}]^{13}$	$4(9), 1.73^{12}$
$I_5(\overline{\overline{I_5}})$	22	3(3)	$[\{-5, 11, 3\}, \{-3, 11, 5\}]^{11}$	$5(6), 1.73^{10}$

Table 2. Bipartite regular graphs satisfying $\sqrt{2} < \lambda_2 \leq \sqrt{3}$.

5. BIPARTITE CUBIC REFLEXIVE GRAPHS

Any regular graph of degree at most 2 (connected or not) is reflexive. Here we determine all bipartite reflexive cubic graphs. Every such a graph must be connected. We start with the following theorem.

Theorem 5. *Every bipartite reflexive cubic graph has at most* 30 *vertices. If any such a graph has exactly* 30 *vertices then its second largest eigenvalue must be equal to* 2.

Proof. Consider any bipartite cubic graph on $2n \geq 30$ vertices, and let G be its bipartite complement. Then G is $(n-3)$-regular and, by Theorem 1, its diameter is 3. Let v be an arbitrary vertex of G, and consider the partition of X with $A = \{v\}$, $B = \{u \in X, d(u, v) = 1\}$, $C = \{u \in X, d(u, v) = 2\}$, $D = \{u \in X, d(u, v) = 3\}$.

We have: $|B| = n - 3$, $|C| = n - 1$, $|D| = 3$. Also, each vertex in D has exactly $n - 3$ neighbours in C, and simple counting shows that there are at least $n - 7$ vertices in C adjacent to all three vertices in D. If C_1 is the set of such $n - 7$ vertices of C, we denote by $C_2 = C \backslash C_1$, so $|C_2| = 6$. Now the partition of X: A, B, C_1, C_2, D induces the quotient matrix (2) with $r = n - 3$, and we get $\lambda_2(G) \geq \lambda_2(Q) = \sqrt{\dfrac{5n - 27}{n - 3}}$. Thus, if $n > 15$ then $\lambda_2(Q) > 2$, while if $n = 15$ then $\lambda_2(Q) = 2$, and the proof follows.

Theorem 6. *The Tutte-Coxeter graph is a unique bipartite reflexive cubic graph on* 30 *vertices. There are no such graphs on* 28 *vertices.*

Proof. Suppose that there is a bipartite reflexive cubic graph on $2n$ ($n \in \{14, 15\}$) vertices, and let G be its bipartite complement. Consider the refinement of the partition of X described in the previous theorem: $A = \{v\}$, $B = \{u \in X, d(u, v) = 1\}$, $D = \{u \in X, d(u, v) = 3\}$, $C_i = \{u \in X, d(u, v) = 2, u$ has exactly i neighbours in $D\}$, where $i \in \{0, 1, 2, 3\}$, and let $|C_i| = c_i$. We have $|D| = 3$, and same as in the previous theorem, $c_3 \geq n - 7$.

If $n = 15$, by Theorem 5, $\lambda_2(G) = 2$, and we also have $|B| = 12$ and $\sum_{i=0}^{3} c_i = 14$. Using the last equality and counting the edges starting from $\bigcup_{i=0}^{3} C_i$, we get $12c_0 + 11c_1 + 10c_2 + 9c_3 = 132$, and consequently $3c_0 + 2c_1 + c_2 = 6$.

The partition of X: A, B, $\bigcup_{i=0}^{2} C_i$, C_3, D induces a quotient matrix whose second largest eigenvalue is $\frac{\sqrt{6}}{2}\sqrt{\frac{24 - c_3}{14 - c_3}}$. Now, $\frac{\sqrt{6}}{2}\sqrt{\frac{24 - c_3}{14 - c_3}} \leq 2$ gives $c_3 \leq 8$, and since in this case c_3 is at least 8, we get $c_3 = 8, c_2 = 6$, and $c_0 = c_1 = 0$. This means that for each vertex v of G there are exactly six vertices with ten neighbours in $N(v)$, and exactly eight vertices with nine neighbours in $N(v)$. Thus, if (1) is the adjacency matrix of G, we have the following decomposition:

$$NN^T = 12I + 10M + 9(J - I - M) = 3I + 9J + M, \tag{3}$$

where M is the adjacency matrix of some 6-regular graph on 15 vertices, while I and J denote the identity and all-1 matrices of size 15×15, respectively.

Since $\lambda_2(NN^T) = \lambda_2^2(G) = 4$, using the equation (3), we get $\lambda_2(M) = 1$. According to [**9**, Theorem 3.4], there are seven 6-regular graphs on 15 vertices with $\lambda_2 = 1$. The matrix NN^T has non-negative spectrum, and this property fails to hold for 6 out of 7 candidates for matrix M. The seventh one pases this test: it is a strongly regular graph with spectrum $[6, 1^9, -3^5]$. Thus, the spectrum of G would be $[\pm 12, \pm 2^9, 0^{10}]$, and the spectrum of its bipartite complement $[\pm 3, \pm 2^9, 0^{10}]$. There is a unique graph with this spectrum, known as the Tutte-Coxeter graph [**13**, p. 115].

If $n = 14$ then $|B| = 11$, and $\sum_{i=0}^{3} c_i = 13$. Already described partition of X induces a quotient matrix whose second largest eigenvalue is in this case equal to $\frac{1}{2}\sqrt{\frac{1452 - 60c_3}{143 - 11c_3}}$ which, together with $c_3 \geq 7$, yields $c_3 = 7$. We also get a unique possibility for the remaining parameters: $c_2 = 6$, and $c_0 = c_1 = 0$. Same as in the previous part of the proof, we get $NN^T = 11I + 9M + 8(J - I - M)$, where M is the adjacency matrix of some 6-regular graph on 14 vertices with $\lambda_2(M) = 1$. According to [**9**, Theorem 3.4], the only such a graph is $\overline{\overline{7K_2}}$, but it does not produce any solution (since in this case NN^T does not have non-negative spectrum). □

Using computer search we generated exactly 280250 connected bipartite cubic graphs on at most 26 vertices. Those with $\sqrt{3} < \lambda_2 \leq 2$ are given in the following theorem.

Theorem 7. *There are exactly* 17 *bipartite cubic graphs satisfying* $\sqrt{3} < \lambda_2 \leq 2$. *They are listed in Table* 3.

Combining the results given in the previous two sections, summarized in Theorem 3 and Theorem 4, as well as those in Theorem 7, we get the following result.

Theorem 8. *There are exactly* 24 *bipartite reflexive cubic graphs. These are* $K_{3,3}, \overline{\overline{4K_2}}$, *the Heawood graph,* $\overline{\overline{C_{10}}}$, *the graphs* I_i $(i = 1, 2, 3)$ *of Table* 2, *and the graphs of Table* 3.

graph	$2n$	diam	LCF notation	spectrum (non-negative part)
J_1	10	3	$[5,-3,-3,3,3]^2$	$3, 2, 1^2, 0^2$
J_2	12	4	$[-3,3]^6$	$3, 2^2, 1, 0^4$
J_3	12	4	$[-5,-5,3,-5,-5,-3;-]$	$3, 2, 1.41^2, 1, 0^2$
J_4	14	4	$[5,-3,5,7,5,-5,5;-]$	$3, 2, 1.41^4, 0^2$
J_5	14	4	$[7,7,-5,3,5,7,-3]^2$	$3, 1.93^2, 1.41^2, 0.52^2$
J_6	16	4	$[7,-5,3,-5;-]^2$	$3, 2^2, 1.73^2, 1, 0^4$
J_7	16	4	$[-5,7,-5,7,-5,7,3,7;-]$	$3, 2, 1.73^2, 1.41^2, 1, 0^2$
J_8	18	4	$[5,7,-5,7,9,-5,5,7,-7;-]$	$3, 2, 1.73^4, 1^2, 0^2$
J_9	18	4	$[5,-5]^9$	$3, 1.97^2, 1.73^2, 1.29^2, 0.68^2$
J_{10}	20	5	$[5,-5,9,-9]^5$	$3, 2^4, 1^5$
J_{11}	20	5	$[5,-9,-7,7,-7,-5,5,-9,5,7;-]$	$3, 2^4, 1^5$
J_{12}	20	5	$[-5,9,-9,5,-5,7,-9,7,-5,9;-]$	$3, 2^3, 1.73^2, 1^3, 0^2$
J_{13}	20	5	$[5,9,-9,5,-9,-5,-9,5,-5,9;-]$	$3, 2^2, 1.88^2, 1.53^2, 1, 0.35^2$
J_{14}	20	5	$[5,9,-5,9,-9,-5,5,9,5,9;-]$	$3, 2^3, 1.73^2, 1^3, 0^2$
J_{15}	20	4	$[-9,5,-7,7,-7,7,-5,7,-9,7;-]$	$3, 2^2, 1.88^2, 1.53^2, 1, 0.35^2$
J_{16}	24	4	$[5,-9,7;-]^4$	$3, 2^6, 1^3, 0^4$
T-Cox	30	4	$[-13,-9,7;-]^5$	$3, 2^9, 0^{10}$

Table 3. Bipartite cubic graphs satisfying $\sqrt{3} < \lambda_2 \leq 2$.

6. CONCLUDING REMARKS

Recall that all r-regular graphs $(r \leq 4)$ with $\lambda_2 \leq 1$ are determined in [**14**], and so this reference contains all graphs of Table 1 that satisfy the same spectral property. The Clebsch graph appears in [**9**] as one of the 5-regular graphs with $\lambda_2 \leq 1$.

All bipartite distance-regular graphs with $\sqrt{2} < \lambda_2 \leq \sqrt{3}$ are determined in [**15**]. All of them are given in Table 2 under identifications $C_{10}, C_{12}, I_3, I_4, \overline{\overline{I_4}}, I_5$, and $\overline{\overline{I_5}}$. In this paper we applied a different approach to determine all such (not only distance regular) graphs.

Note that there are no non-bipartite triangle-free regular graphs with $\lambda_2 = \sqrt{2}$. In fact, this equality holds only for 3 bipartite regular graphs (see Table 1). In addition, there are 11 bipartite regular graphs with $\lambda_2 = \sqrt{2}$ (Table 2), and 15 bipartite cubic graphs with $\lambda_2 = 2$ (Table 3).

The graphs I_1 (Table 1), J_2, and J_3 (Table 3) are isomorphic to their bipartite complements, while the bipartite complements of the remaining 21 bipartite reflexive cubic graphs are also bipartite reflexive but not cubic.

Some of the graphs obtained appear in the literature. For example, I_1, I_2, I_3 (Table 2), J_{10} and J_{16} (Table 3) are known as Franklin, Möbius-Kantor, Pappus, Desargues, and Nauru graph, respectively.

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REFERENCES

1. H. S. M. Coxeter, R. Frucht, D. L. Powers: *Zero-symmetric graphs: trivalent graphical regular representations of groups.* Academic Press. New York, 1981.
2. D. Cvetković, M. Doob, H. Sachs: *Spectra of graphs – theory and application.* 3rd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, 1995.
3. D. Cvetković, S. Simić: *On graphs whose second largest eigenvalue does not exceed* $\frac{\sqrt{5}-1}{2}$. Discrete Math., **138** (1995), 213–227.
4. D. Cvetković, S. Simić: *The second largest eigenvalue of a graph* (*a survey*). Filomat, **9** (1995), 449–472.
5. P. Erdős, S. Fajtlowits, W. Staton: *Degree sequences in triangle-free graphs.* Discrete Math., **92** (1991), 85–88.
6. R. Feria-Purón, G. Pineda-Villavicencio: *On bipartite graphs of defect at most* 4. Discrete Appl. Math., **160** (2012), 140–154.
7. W. H. Haemers: *Eigenvalue techniques in design and graph theory.* (Ph.D. thesis Technical University Eindhoven 1979.), Math. Centre Tract 121, Mathematical Centre, Amsterdam, 1980.
8. S. Hoory, N. Linial, A. Wigderson: *Expander graphs and their applications.* Bull. Amer. Math. Soc., **43** (2006), 439–561.
9. T. Koledin, Z. Stanić: *Regular graphs whose second largest eigenvalue is at most* 1. Novi Sad J. Math., **43(1)** (2013), 145–153.
10. T. Koledin, Z. Stanić: *Some spectral inequalities for triangle-free regular graphs.* Filomat, in press.
11. M. Meringer: *Fast generation of regular graphs and construction of cages.* J. Graph Theory, **30** (1999), 137–146.
12. M. Hall, Jr.: *Combinatorial Theory.* 2nd edition, John Wiley and Sons Inc., New York, 1986.
13. M. Petrović, Z. Radosavljević: *Spectrally constrained graphs.* Faculty of Science, Kragujevac, Yugoslavia, 2001.
14. Z. Stanić: *On regular graphs and coronas whose second largest eigenvalue does not exceed* 1. Linear Multilinear Algebra, **58** (2010), 545–554.

15. Y. Teranishi, F. Yasuno: *The second largest eigenvalues of regular bipartite graphs.* Kyushu J. Math., **54** (2000), 39–54.

Faculty of Electrical Engineering,
University of Belgrade,
Bulevar kralja Aleksandra 73,
11 000 Belgrade
Serbia
E-mail: tamara@etf.rs

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Faculty of Mathematics,
University of Belgrade,
Studentski trg 16,
11 000 Belgrade
Serbia
E-mail: zstanic@math.rs; zstanic@matf.bg.ac.rs