Desanka Radunović – NUMERIČKE METODE

Linearna algebra; jednačine

Matrice

$$A = (a_{ij}), \qquad A^* = (\overline{a}_{ji}), \qquad A^\top = (a_{ji})$$

Hermite-ova $\quad A^* = A \quad (A^\top = A \quad$ simetrična)

unitarna $\quad A^* = A^{-1} \quad (A^\top = A^{-1} \quad$ ortogonalna)

Norme vektora i indukovane norme matrica $\quad \|A\| = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$

uniformna
$$\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|, \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

apsolutna
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|, \qquad \|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$$

euklidska
$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}, \qquad \|A\|_2 = \sqrt{\max_{1\le i\le n} \lambda_i(A^*A)}$$

sferna $\qquad \|A\|_s = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|^2}, \qquad \|A\mathbf{x}\|_2 \le \|A\|_s\,\|\mathbf{x}\|_2$

uslovljenost

$$\text{cond}(I) = 1 \le \text{cond}(A) = \|A\| \cdot \|A^{-1}\| \le \infty,$$

$$\text{cond}(A) = \infty \quad \text{za} \quad \det(A) = 0$$

Sopstvene vrednosti i vektori

$$A\,\mathbf{x} = \lambda\,\mathbf{x} \qquad \Longleftrightarrow \qquad D(\lambda) \equiv \det(A - \lambda\, I) = 0$$

$$|\lambda|\,\|\mathbf{x}\| = \|\lambda\mathbf{x}\| = \|A\,\mathbf{x}\| \le \|A\|\,\|\mathbf{x}\| \qquad \longrightarrow \qquad |\lambda| \le \|A\|$$

$$A^* = A, \qquad \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,$$

$$\longrightarrow \qquad \lambda_1 = \max_{\mathbf{x}\neq 0}\frac{(A\mathbf{x}\,,\,\mathbf{x})}{(\mathbf{x}\,,\,\mathbf{x})}, \qquad \lambda_n = \min_{\mathbf{x}\neq 0}\frac{(A\mathbf{x}\,,\,\mathbf{x})}{(\mathbf{x}\,,\,\mathbf{x})}$$

Gauss-ova metoda eliminacije

$$\boxed{A\mathbf{x} = \mathbf{b} \qquad \Longleftrightarrow \qquad U\mathbf{x} = \mathbf{c}} \qquad U = \begin{pmatrix} * & \cdots & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

$$(U\,;\,\mathbf{c}) = L_{n-1}P_{n-1}L_{n-2}P_{n-2}\cdots L_1P_1(A\,;\,\mathbf{b}), \qquad L = L_1^{-1}\cdots L_{n-1}^{-1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$l_{ij} = \frac{\bar{a}_{ij}^{(j-1)}}{\bar{a}_{jj}^{(j-1)}}, \quad L_j = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{j+1,j} & 1 & & \\ & & & & \ddots & \\ 0 & & -l_{n,j} & 0 & & 1 \end{pmatrix}, \qquad P_j = \begin{pmatrix} 0 & & & 1 & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ 1 & & & 0 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

$$x_n = \frac{c_n}{u_{nn}}, \qquad x_i = \frac{1}{u_{ii}}\Big(c_i - \sum_{j=i+1}^{n} u_{ij}\,x_j\Big), \quad i = n-1,\ldots,1.$$

Trougaona dekompozicija matrice $\boxed{A = L\,U}$

$$u_{1k} = a_{1k}, \qquad u_{ik} = a_{ik} - \sum_{j=1}^{i-1} l_{ij} u_{jk}, \quad k = i, \ldots, n$$

$$i = 2, \ldots, n.$$

$$l_{k1} = \frac{a_{k1}}{u_{11}}, \qquad l_{ki} = \frac{1}{u_{ii}} \Big(a_{ki} - \sum_{j=1}^{i-1} l_{kj} u_{ji} \Big), \quad k = i+1, \ldots, n,$$

Cholesky dekompozicija $\boxed{A = L\,L^*, \qquad A^* = A}$

$$l_{11} = \sqrt{a_{11}}, \qquad l_{i1} = \frac{a_{i1}}{l_{11}},$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} |l_{ik}|^2}, \qquad l_{ij} = \frac{1}{l_{jj}} \Big(a_{ij} - \sum_{k=1}^{i-1} l_{ik} \bar{l}_{jk} \Big),$$

$$1 < j < i \le n.$$

Householder-ova matrica

$$\boxed{H = I - 2\,\mathbf{w}\,\mathbf{w}^*, \qquad \|\mathbf{w}\|_2 = 1}$$

osobine

$$H = H^* = H^{-1} \qquad \text{ili} \qquad H^2 = I$$

$$\mathbf{y} = H\,\mathbf{x} \qquad \longrightarrow \qquad \|\mathbf{y}\|_2 = \|\mathbf{x}\|_2, \qquad (\mathbf{y},\,\mathbf{x}) = (\mathbf{x},\,\mathbf{y})$$

$$\mathbf{x} = (x_1, \ldots, x_m)^\top, \quad \sigma = \|\mathbf{x}\|_2, \quad x_1 = |x_1| e^{\imath\alpha}$$

$$\mathbf{y} = k\,\mathbf{e}_1 = (k, 0, \ldots, 0)^\top = H\,\mathbf{x}$$

$$\longrightarrow \qquad k = -\,\sigma e^{\imath\alpha}, \qquad \mathbf{w} = \frac{\mathbf{x} - k\mathbf{e}_1}{\|\mathbf{x} - k\mathbf{e}_1\|_2}$$

Q R dekompozicija

$$\boxed{A = Q\,R}$$

$$R = \begin{pmatrix} * & \cdots & * \\ & \ddots & \\ 0 & & * \end{pmatrix} = T_{n-1} \ldots T_1\,A,$$

$$Q = T_1 \ldots T_{n-1}, \qquad Q^* = Q^{-1} = T_{n-1} \ldots T_1$$

$$T_m = \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \ddots & 0 \\ & \cdots & & & \cdots & \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & H_m & \\ 0 & \cdots & 0 & & & \end{array}\right)$$

$$\text{dim}(H_m) = n{-}m{+}1, \quad \mathbf{x}_m = \left(a_{m,m}^{(m-1)} \quad \ldots \quad a_{n,m}^{(m-1)}\right)^\top \quad \left(A_m = \left\{a_{i,j}^{(m)}\right\} = T_m\,A_{m-1},\ A_0 \equiv A\right)$$

Singularna dekompozicija (SVD)

$$\boxed{A = U\,W\,V^*}$$

$$U^* = U^{-1}, \quad V^* = V^{-1}, \qquad W = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{pmatrix}$$

singularne vrednosti $\qquad \sigma_k^2 = \lambda_k(A^*A), \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$

$$\dim(A) = m \times n, \quad r \leq \min(m,n), \qquad \dim(U) = m \times m, \quad \dim(V) = n \times n,$$

$$(A^*A)\,V = V\,(W^*\,W) \qquad \longrightarrow \qquad V = \left(\mathbf{v}_1, \ldots, \mathbf{v}_n\right), \quad (A^*A)\,\mathbf{v}_k = \sigma_k^2\,\mathbf{v}_k$$

$$k = 1, \ldots r,$$

$$(A\,A^*)\,U = U\,(W\,W^*) \qquad \longrightarrow \qquad U = \left(\mathbf{u}_1, \ldots, \mathbf{u}_m\right), \quad (A\,A^*)\,\mathbf{u}_k = \sigma_k^2\,\mathbf{u}_k$$

$$r = n = \dim(A) \quad \longrightarrow \quad \sigma_1 = \|A\|_2, \qquad \sigma_n = \frac{1}{\|A^{-1}\|_2}, \qquad \mathrm{cond}(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$$

sistem

LU $\quad L\,\mathbf{y} = P\,\mathbf{b}, \qquad U\,\mathbf{x} = \mathbf{y} \qquad (P\,A\,\mathbf{x} = L\,U\,\mathbf{x} = P\,\mathbf{b})$

Cholesky $\quad L\,\mathbf{y} = \mathbf{b}, \qquad L^*\,\mathbf{x} = \mathbf{y}$

QR $\quad \mathbf{y} = Q^*\mathbf{b}, \qquad R\,\mathbf{x} = \mathbf{y}$

SVD $\quad \mathbf{x} = V\,W^{-1}\,U^*\mathbf{b}$

determinanta

LU $\quad \det(P\,A) = \pm\det(A) = \det(L)\cdot\det(U) = u_{11}\cdots u_{nn}$

Cholesky $\quad \det(A) = (l_{11}\cdots l_{nn})^2$

inverzna matrica $\quad A\,A^{-1} = I \qquad \longrightarrow \qquad A\,\mathbf{x}_i = \mathbf{e}_i, \quad i = 1,\ldots,n.$

$$A^{-1} = (\mathbf{x}_1,\ldots,\mathbf{x}_n)\,, \qquad I = (\mathbf{e}_1,\ldots,\mathbf{e}_n)$$

Iterativne metode $F(x) = 0 \iff x = G(x)$

$$x^{(k+1)} = G(x^{(k)}), \quad k = 0, 1, \ldots, \qquad \lim_{k\to\infty} x^{(k)} = \overline{x}, \quad \overline{x} = G(\overline{x})$$

kontrakcija $\|G(x) - G(y)\| \le q\|x - y\|, \qquad 0 \le q < 1$

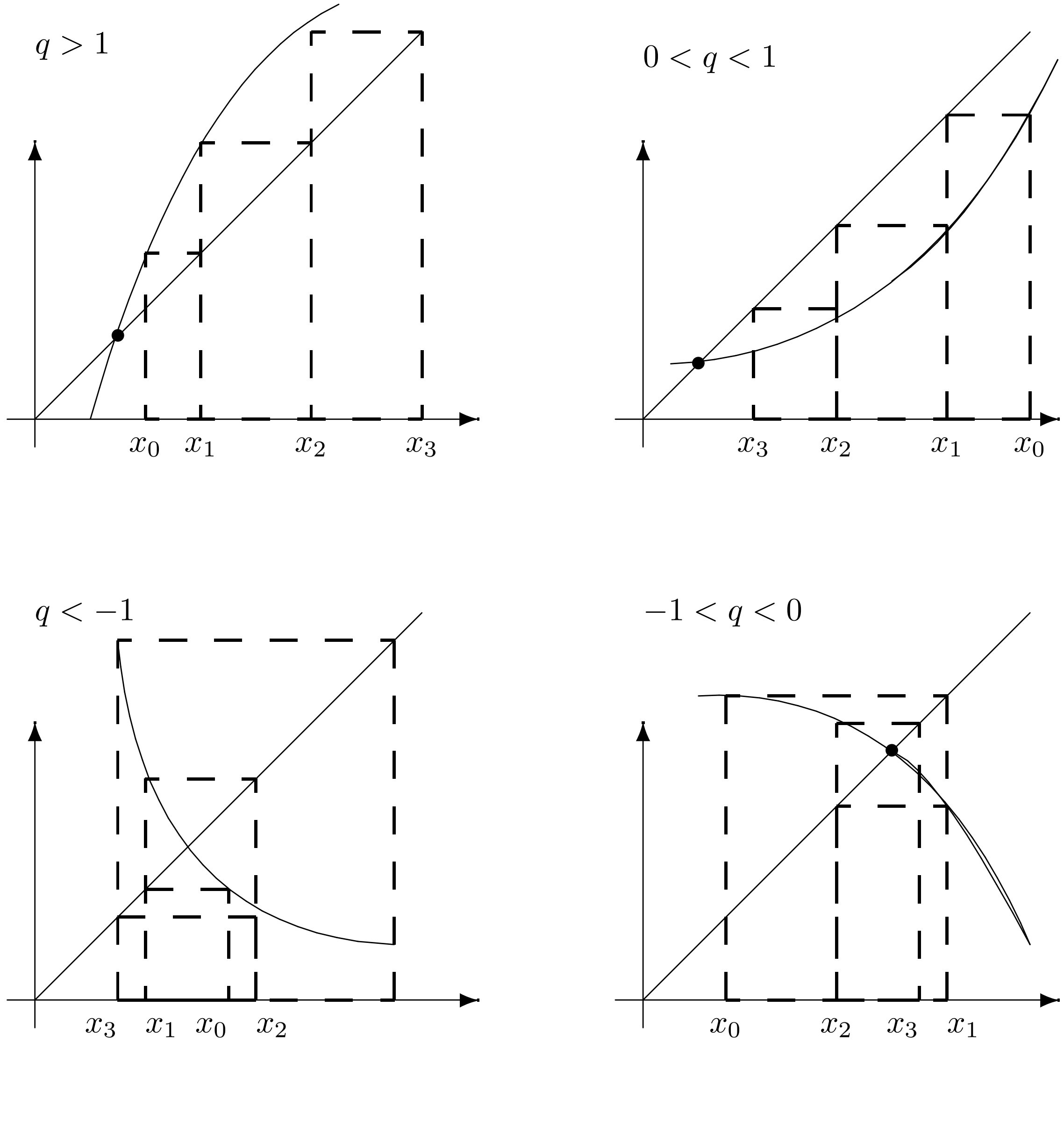
$$\mathbf{x} \in \mathcal{R}^n \qquad q = \max_z \|J\big(G(\mathbf{z})\big)\|, \qquad J\big(G(\mathbf{z})\big) = \left\{\frac{\partial g_i}{\partial x_j}(\mathbf{z})\right\}_{i,j=1}^{n}$$

greška

apriorna $\|\overline{x} - x^{(k)}\| \le \frac{q^k}{1-q}\|G(x^{(0)}) - x^{(0)}\| \longrightarrow k \ge \frac{\ln \frac{\varepsilon(1-q)}{\|G(x^{(0)})-x^{(0)}\|}}{\ln q}$

aposteriorna $\|\overline{x} - x^{(k)}\| \le \frac{q}{1-q}\|x^{(k)} - x^{(k-1)}\| \longrightarrow \|x^{(k)} - x^{(k-1)}\| \le \frac{1-q}{q}\,\varepsilon$

Geometrijska interpretacija metode iteracije $x^{(k+1)} = g(x^{(k)}), \quad (n = 1)$



”Preconditioning” (preduslovljavanje) matricom P

$$A\,\mathbf{x} = \mathbf{b} \qquad \Longleftrightarrow \qquad P\,\mathbf{x} = N\,\mathbf{x} + \mathbf{b} \qquad \text{za} \quad A = P - N$$

$\mathbf{x}^{(0)}$ proizvoljno $\qquad P\,\mathbf{x}^{(k+1)} = N\,\mathbf{x}^{(k)} + \mathbf{b}, \qquad k = 0, 1 \ldots,$

$$\boxed{\mathbf{x}^{(k+1)} = P^{-1}N\,\mathbf{x}^{(k)} + P^{-1}\mathbf{b}}$$

konvergencija $\qquad J\big(G(\mathbf{x})\big) \equiv P^{-1}N \quad \longrightarrow \quad \|P^{-1}N\| < 1$

za $\quad N = P - A \qquad \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - P^{-1}(A\,\mathbf{x}^{(k)} - \mathbf{b})$

vektor greške $\quad \mathbf{r}^{(k)} = A\,\mathbf{x}^{(k)} - \mathbf{b},$ $\qquad \boxed{\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - P^{-1}\,\mathbf{r}^{(k)}}$

Metode, $A = L + D + U$ (donji trougao + dijagona + gornji trougao),

Jacobi

$$P = D, \qquad N = -(L + U)$$

$$\mathbf{x}^{(k+1)} = -D^{-1}(L + U)\,\mathbf{x}^{(k)} + P^{-1}\,\mathbf{b} = \mathbf{x}^{(k)} - D^{-1}\,\mathbf{r}^{(k)}$$

Gauss-Seidel

$$P = L + D, \qquad N = -U$$

$$\mathbf{x}^{(k+1)} = -(L + D)^{-1}U\,\mathbf{x}^{(k)} + (L + D)^{-1}\,\mathbf{b} = \mathbf{x}^{(k)} - (L + D)^{-1}\,\mathbf{r}^{(k)}$$

Relaksacija

$$\omega\, A\,\mathbf{x} = \omega\,\mathbf{b}, \qquad 0 < \omega < 2 \qquad (\omega = 1 \quad \text{G-S})$$

$$P = \omega\, L + D, \qquad N = -\big(\omega\, U + (\omega - 1)\, D\big)$$

$$\mathbf{x}^{(k+1)} = -(\omega\, L + D)^{-1}\big(\omega\, U + (\omega - 1)\, D\big)\,\mathbf{x}^{(k)} + \omega\,(\omega\, L + D)^{-1}\,\mathbf{b}$$
$$= \mathbf{x}^{(k)} - \omega\,(\omega\, L + D)^{-1}\,\mathbf{r}^{(k)}$$

Richardson-ova metoda (uopštenje uvođenjem parametra τ_k)

stacionarna $$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \tau\, P^{-1}\, \mathbf{r}^{(k)} \qquad (\tau_k \equiv \tau)$$

$$P^{-1} = I \qquad \longrightarrow \qquad \tau_{\text{opt}} = \frac{2}{M+m}, \qquad M = \max_i |\lambda_i(A)|, \quad m = \min_i |\lambda_i(A)|$$

$$\|\bar{\mathbf{x}} - \mathbf{x}^{(k)}\|_2 \le \left(\frac{M-m}{M+m}\right)^k \|\bar{\mathbf{x}} - \mathbf{x}^{(0)}\|_2$$

nestacionarna $$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \tau_k\, P^{-1}\, \mathbf{r}^{(k)}$$

$$P^{-1} = I \qquad \longrightarrow \qquad \tau_{j,\text{opt}} = \frac{2}{M+m+(M-m)\cos\frac{2j-1}{2k}\pi}, \quad j = 1,\ldots,k,$$

$$\|\bar{\mathbf{x}} - \mathbf{x}^{(k)}\|_2 \le 2\left(\frac{\sqrt{M}-\sqrt{m}}{\sqrt{M}+\sqrt{m}}\right)^k \|\bar{\mathbf{x}} - \mathbf{x}^{(0)}\|_2,$$

Gradijentne metode

$$A = A^*, \qquad A\,\bar{\mathbf{x}} = \mathbf{b} \qquad \Longleftrightarrow \qquad F(\bar{\mathbf{x}}) = \min\, F(\mathbf{x})$$

funkcional
$$F(\mathbf{x}) = \frac{1}{2}\,(A\,\mathbf{x},\,\mathbf{x}) - (\mathbf{b},\,\mathbf{x})$$

gradijent
$$\nabla F(\mathbf{x}^{(k)}) = \begin{pmatrix} \frac{\partial F}{\partial x_1} & \dots & \frac{\partial F}{\partial x_n} \end{pmatrix}^\top = A\,\mathbf{x}^{(k)} - \mathbf{b} = \mathbf{r}^{(k)}$$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \lambda_k\,\mathbf{d}^{(k)}, \quad k = 0, 1, \dots, \qquad \lim_{k\to\infty} \mathbf{x}^{(k)} = \bar{\mathbf{x}}$$

popravka
$$\lambda_k = -\frac{\left(\nabla F(\mathbf{x}^{(k)}),\,\mathbf{d}^{(k)}\right)}{\left(A\,\mathbf{d}^{(k)},\,\mathbf{d}^{(k)}\right)} = -\frac{\sum_{i=1}^n \frac{\partial F(\mathbf{x}^{(k)})}{\partial x_i}\,d_i^{(k)}}{\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 F(\mathbf{x}^{(k)})}{\partial x_i\,\partial x_j}\,d_i^{(k)}\,d_j^{(k)}}$$

pokoordinatni spust $\mathbf{d}^{(k)} = \mathbf{e}^{(k)}, \qquad \mathbf{e}^{(k)} = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}^\top$

najstrmiji spust $\mathbf{d}^{(k)} = -\nabla F(\mathbf{x}^{(k)}) = -\,\mathbf{r}^{(k)}, \quad \lambda_k = -\delta_k$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \delta_k\,\mathbf{r}^{(k)}, \qquad \delta_k = \frac{\left(\mathbf{r}^{(k)},\ \mathbf{r}^{(k)}\right)}{\left(A\,\mathbf{r}^{(k)},\ \mathbf{r}^{(k)}\right)}, \qquad k = 0, 1, \ldots,$$

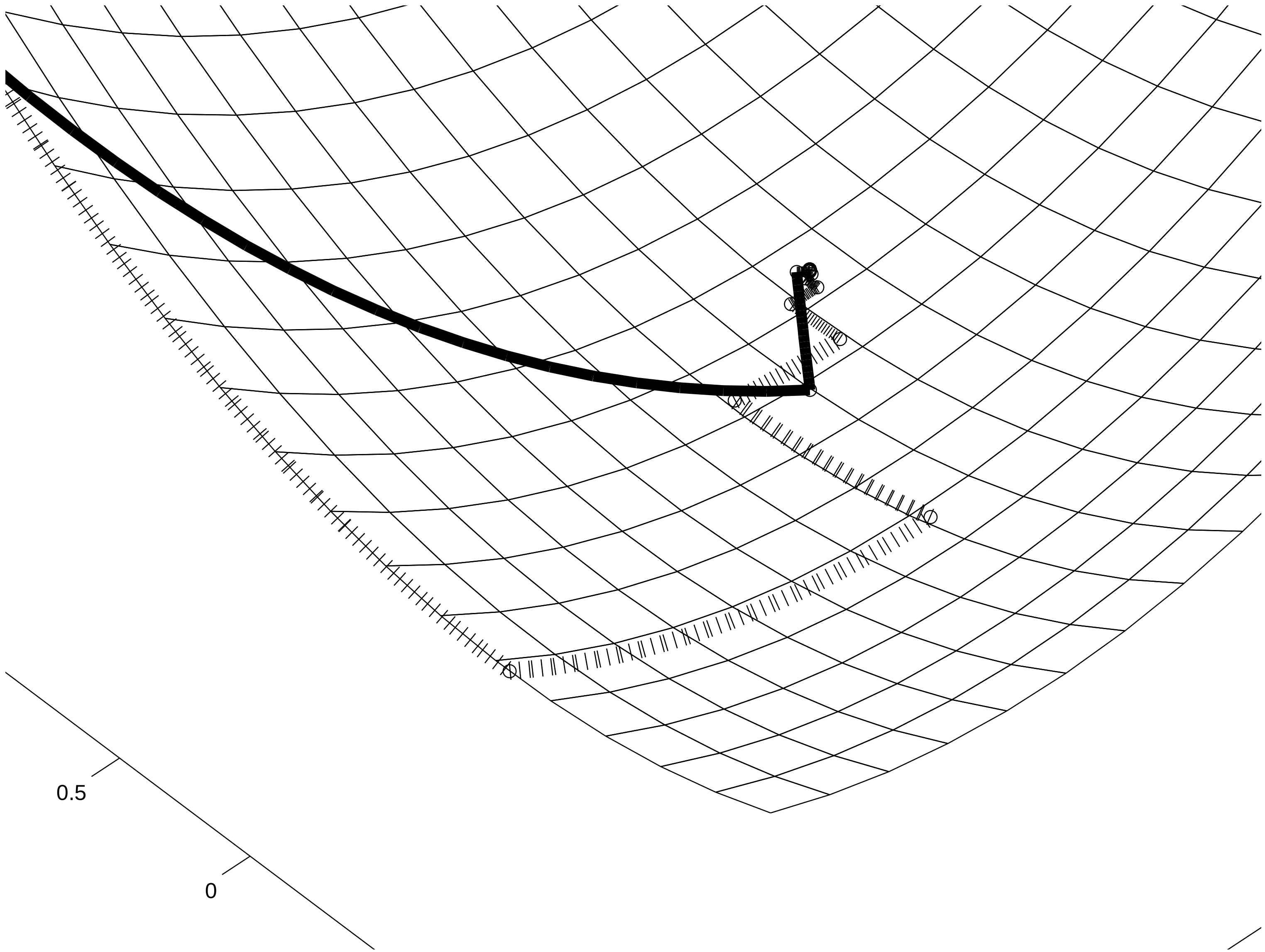
konjugovani gradijent $\mathbf{d}^{(k)} = \mathbf{p}^{(k)}, \qquad \left(A\mathbf{p}^{(i)}, \mathbf{p}^{(j)}\right) = \delta(i-j)$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \lambda_k\,\mathbf{p}^{(k)}, \qquad k = 0, 1, \ldots,$$

$$\mathbf{r}^{(k)} = A\,\mathbf{x}^{(k)} - \mathbf{b}, \qquad \mathbf{p}^{(k+1)} = \mathbf{r}^{(k+1)} - \mu_k\,\mathbf{p}^{(k)}, \quad \mathbf{p}^{(0)} = \mathbf{r}^{(0)}$$

$$\lambda_k = \frac{\left(\mathbf{r}^{(k)},\ \mathbf{p}^{(k)}\right)}{\left(A\mathbf{p}^{(k)},\ \mathbf{p}^{(k)}\right)}, \qquad \mu_k = \frac{\left(A\mathbf{p}^{(k)},\ \mathbf{r}^{(k+1)}\right)}{\left(A\mathbf{p}^{(k)},\ \mathbf{p}^{(k)}\right)}$$

Pokoordinatni i najbrži spust



Metoda najmanjih kvadrata

$$\min_{\mathbf{x}} \|\mathbf{r}(\mathbf{x})\|_2$$

$$\|\mathbf{r}(\mathbf{x})\|_2{}^2 = \|A\,\mathbf{x} - \mathbf{b}\|_2{}^2 = (A^\top A\,\mathbf{x}, \mathbf{x}) - 2(A^\top \mathbf{b}, \mathbf{x}) + (\mathbf{b}, \mathbf{b})$$

Pridruženi problem sa samokonjugovanim operatorom

$$A^\top \not= \; A\,\mathbf{x} = \mathbf{x} \qquad \longrightarrow \qquad A^\top A\,\mathbf{x} = A^\top \mathbf{x}$$

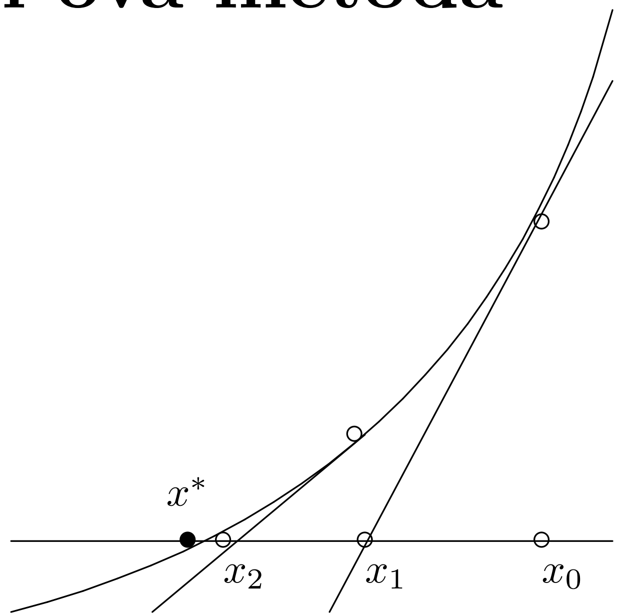
$$F(\mathbf{x}) = \frac{1}{2}(A^\top A\,\mathbf{x}, \mathbf{x}) - (A^\top \mathbf{b}, \mathbf{x})$$

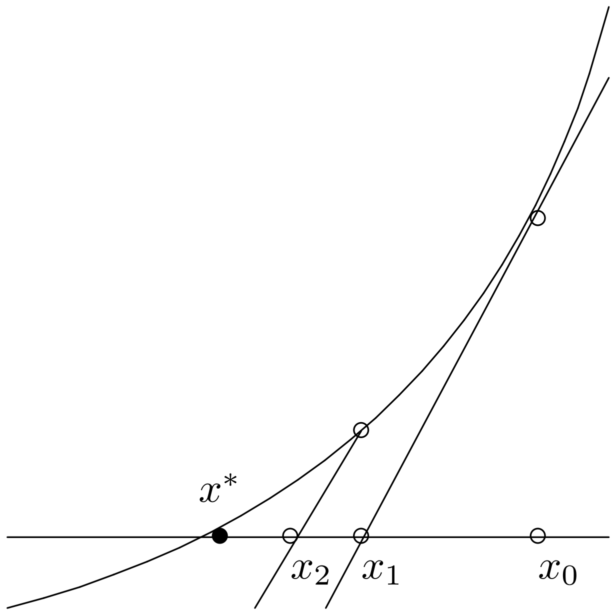
$$A\,\overline{\mathbf{x}} = \mathbf{b} \qquad \Longleftrightarrow \qquad \begin{array}{c} \|\mathbf{r}(\overline{\mathbf{x}})\|_2 = \min \|\mathbf{r}(\mathbf{x})\|_2, \\ F(\overline{\mathbf{x}}) = \min F(\mathbf{x}) \end{array}$$

Omogućava primenu gradijentnih metoda i kada matrica sistema nije Hermite-ova

$$f(x) = 0, \qquad x \in \mathcal{R}^1$$

Newton-ova metoda

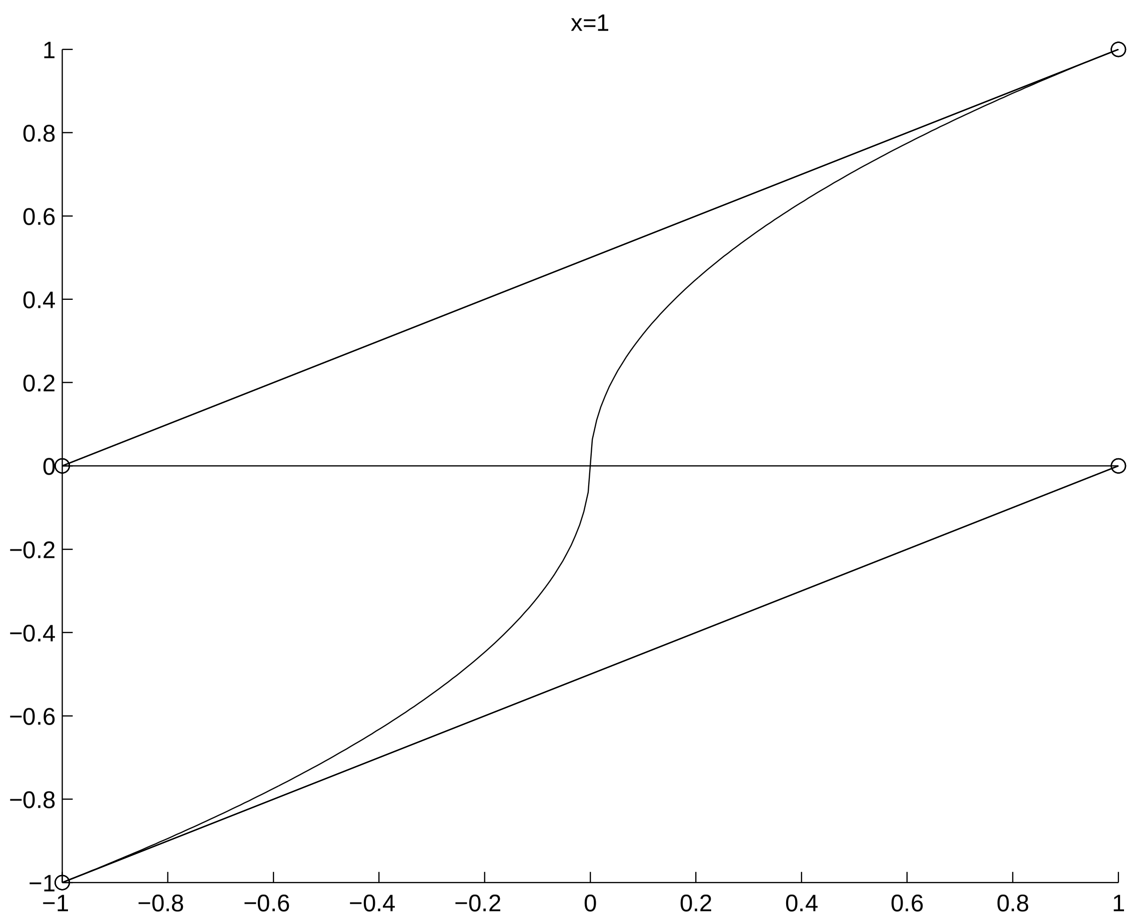




$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f'\left(x^{(k)}\right)},$$

$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f'\left(x^{(0)}\right)}$$

greška $$|\overline{x} - x^{(k)}| \le \frac{M_2}{2m_1}|x^{(k)} - x^{(k-1)}|^2$$



$m_1 = \min |f'(x)|$

$M_2 = \max |f''(x)|$

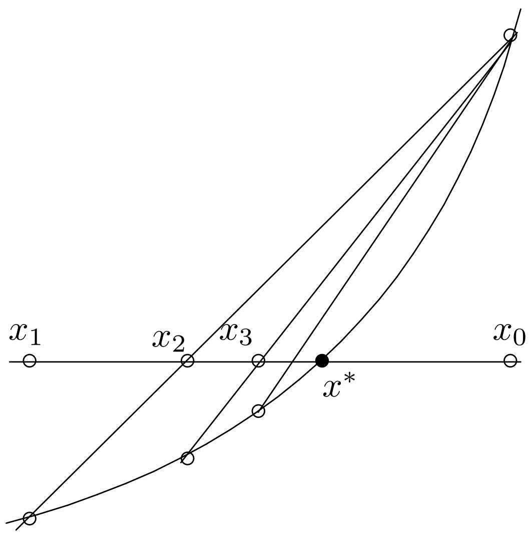
Uslovi konvergencije:

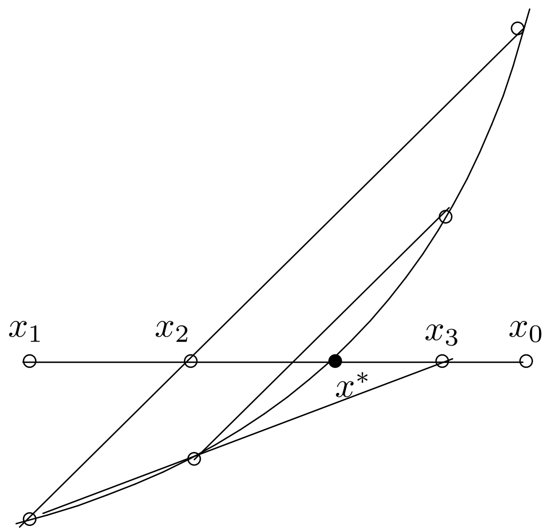
- $f \in \mathcal{C}^1[a,b]$
- $f(a)f(b) < 0$
- sign$f'(x) =$ const, $f'(x) \neq 0$
- sign$f''(x) =$ const
- $f(x_0)f''(x_0) > 0$

Regula-falsi

(levo)

$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f\left(x_F\right) - f\left(x^{(k)}\right)}(x_F - x^{(k)})$$



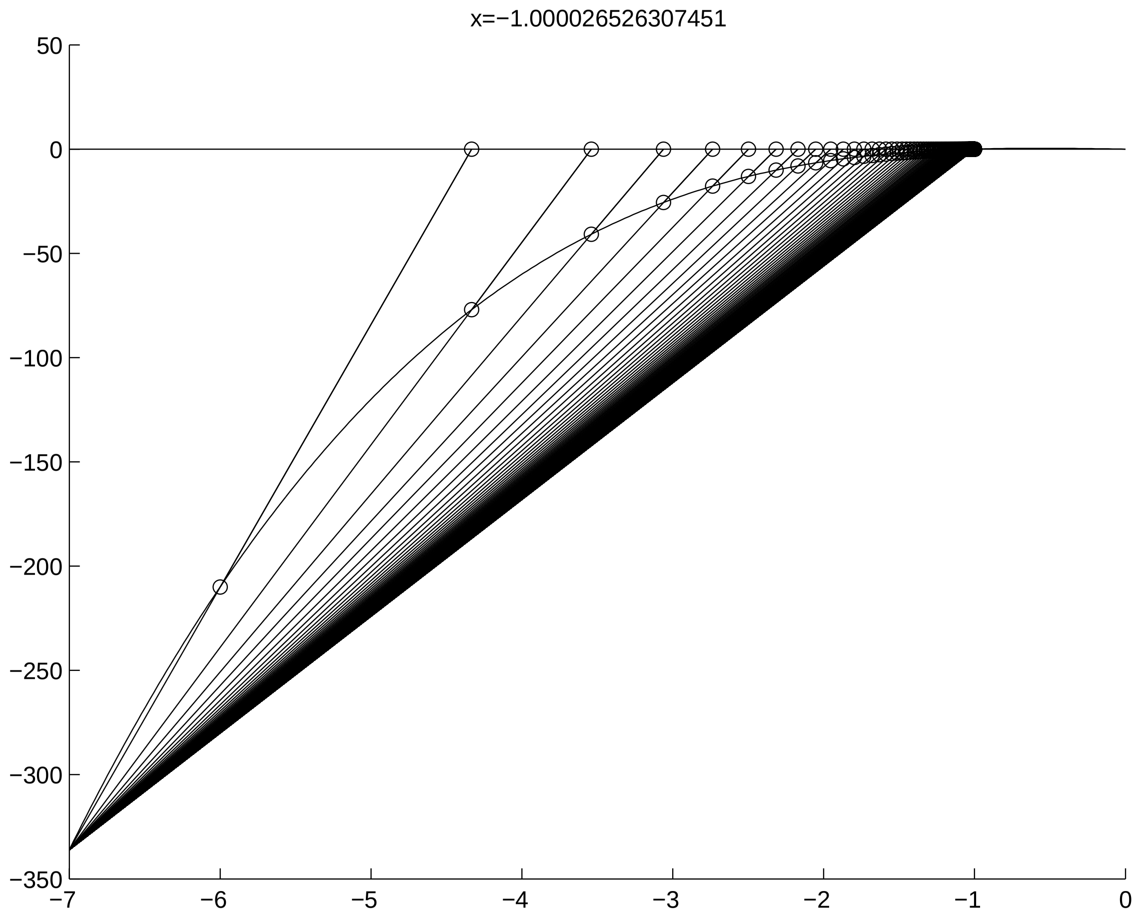


Sečica

$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f\left(x^{(k-1)}\right) - f\left(x^{(k)}\right)}(x^{(k-1)} - x^{(k)})$$

(desno)

Iteracije metode regula-falsi



$$f'(x) \approx \frac{f\left(x_F\right) - f\left(x^{(k)}\right)}{x_F - x^{(k)}}$$

Greška:

$$|\overline{x} - x^{(k)}| \leq \frac{M_1 - m_1}{m_1} |x^{(k)} - x^{(k-1)}|$$

$m_1 = \min |f'(x)|$

$M_1 = \max |f'(x)|$

Metoda polovljenja

$$\overline{x} \approx x^{(k)} = \frac{1}{2}(a_k + b_k),$$

greška $\quad |\overline{x} - x^{(k)}| \le \frac{1}{2^{k+1}}(b-a) \le \varepsilon \quad$ ili $\quad k \ge \left[\frac{\ln \frac{b-a}{\varepsilon}}{\ln 2} \right]$

$$[a,\, b] \equiv [a_0, b_0] \supset \cdots \supset [a_k, b_k] \supset \cdots, \qquad f(a_k) f(b_k) < 0, \qquad b_k - a_k = \frac{1}{2^k}(b-a)$$

Metoda Bairstow-a

$$P_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n = 0$$

$$P_n(x) = (x^2 + px + q)\,(x^{n-2} + b_1 x^{n-3} + \cdots + b_{n-3}x + b_{n-2}) + rx + s$$

$$b_m = a_m - p\,b_{m-1} - q\,b_{m-2}, \quad m = 1, \ldots, n, \qquad b_{-1} = 0, \quad b_0 = 1$$

$$r = a_{n-1} - p\,b_{n-2} - q\,b_{n-3} = b_{n-1}, \qquad s = a_n - q\,b_{n-2} = b_n + p\,b_{n-1}$$

$$c_m = b_m - p\,c_{m-1} - q\,c_{m-2}, \quad m = 1, \ldots, n-1, \qquad c_{-1} = 0, \quad c_0 = 1$$

$$\begin{array}{ll} c_{n-2}\Delta p + c_{n-3}\Delta q = b_{n-1} & \Delta p \equiv \Delta p^{(k)} = p^{(k+1)} - p^{(k)} \\ c'_{n-1}\Delta p + c_{n-2}\Delta q = b_n & \Delta p \equiv \Delta q^{(k)} = q^{(k+1)} - q^{(k)} \end{array}, \quad k = 0, 1, \ldots.$$

Sistemi nelinearnih jednačina

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \qquad x \in \mathcal{R}^n$$

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_n(\mathbf{x}) \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad F'(\mathbf{x}) = J\big(\mathbf{F}(\mathbf{x})\big) = \begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \cdots & \frac{\partial f_n(\mathbf{x})}{\partial x_1} \end{pmatrix}$$

Newton-ova metoda

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \big[F'(\mathbf{x}^{(k)})\big]^{-1}\mathbf{F}\big(\mathbf{x}^{(k)}\big)$$

$$\|F'(\mathbf{x}) - F'(\mathbf{y})\| \leq \gamma\|\mathbf{x} - \mathbf{y}\|$$

$[F'(\mathbf{x})]^{-1}$ postoji i $\|[F'(\mathbf{x})]^{-1}\| \leq \beta$

$$\|[F'\big(\mathbf{x}^{(0)}\big)]^{-1}\mathbf{F}\big(\mathbf{x}^{(0)}\big)\| \leq \alpha$$

$\mathbf{x}$, $\mathbf{y}$ u konveksnom skupu $C \in \mathcal{X}$, $\quad h = \frac{\alpha\beta\gamma}{2} < 1$

$\longrightarrow$

$$\lim_{k\to\infty} \mathbf{x}^{(k)} = \overline{\mathbf{x}}, \quad \mathbf{F}(\overline{\mathbf{x}}) = \mathbf{0}$$

greška

$$\|\overline{\mathbf{x}} - \mathbf{x}^{(k)}\| \leq \alpha\frac{h^{2^k-1}}{1 - h^{2^k}}$$

Metoda iteracije $\quad \mathbf{F}(\mathbf{x}) = \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{x} = \mathbf{G}(\mathbf{x}) \equiv \{g_i(\mathbf{x})\}_{i=1}^{n}$

$$\mathbf{x}^{(0)}, \qquad \mathbf{x}^{(k+1)} = \mathbf{G}\big(\mathbf{x}^{(k)}\big), \quad k = 0, 1, \ldots,$$

$$q = \max_{x} \|J\big(\mathbf{G}(\mathbf{x})\big)\| < 1 \qquad \longrightarrow \qquad \lim_{k\to\infty} \mathbf{x}^{(k)} = \bar{\mathbf{x}}, \quad \bar{\mathbf{x}} = \mathbf{G}\big(\bar{\mathbf{x}}\big)$$

greška $\qquad \|\bar{\mathbf{x}} - \mathbf{x}^{(k)}\| \leq \varepsilon \qquad$ za $\qquad \|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\| \leq \dfrac{1-q}{q}\,\varepsilon \quad$ ili $\quad k \geq \dfrac{\ln \frac{\varepsilon(1-q)}{\|\mathbf{G}\left(\mathbf{x}^{(0)}\right) - \mathbf{x}^{(0)}\|}}{\ln q}$

Gradijentne metode $\quad \mathbf{F}\big(\bar{\mathbf{x}}\big) = \mathbf{0} \quad \Longleftrightarrow \quad F\big(\bar{\mathbf{x}}\big) = \min F(\mathbf{x})$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \lambda_k\,\mathbf{d}^{(k)}, \qquad \lambda_k = -\frac{\nabla F\big(\mathbf{x}^{(k)}\big)\cdot \mathbf{d}^{(k)}}{(\mathbf{d}^{(k)})^\top A\big(\mathbf{x}^{(k)}\big)\,\mathbf{d}^{(k)}}, \qquad k = 0, 1, \ldots .$$

$$F\big(\mathbf{x}^{(k)} + \lambda_k \mathbf{d}^{(k)}\big) \approx F\big(\mathbf{x}^{(k)}\big) + \lambda_k\,\nabla F\big(\mathbf{x}^{(k)}\big)\cdot \mathbf{d}^{(k)} + \tfrac{1}{2}\,\lambda_k^2\,(\mathbf{d}^{(k)})^\top A\big(\mathbf{x}^{(k)}\big)\,\mathbf{d}^{(k)}$$

Sopstvene vrednosti i vektori

$$\boxed{A\,\mathbf{x}_k = \lambda_k\,\mathbf{x}_k}$$

$$D(\lambda_k) = \text{det}(A - \lambda_k\, I) = (-1)^n(\lambda_k^n + p_1\lambda_k^{n-1} + \cdots + p_n) = 0$$

Interpolacija

$$D(\lambda) \equiv \sum_{i=0}^{n} \Bigg(\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-\mu_j}{\mu_i-\mu_j} \Bigg) D(\mu_i), \qquad |\mu_i| \le \|A\|$$

Le Verrier

$$p_k = -\frac{1}{k}(S_k + p_1 S_{k-1} + p_2 S_{k-2} + \cdots + p_{k-1}S_1), \quad k = 2,\ldots,n.$$

$$p_1 = -S_1, \qquad S_k = tr(A^k) = \sum_{i=1}^{n} a_{i,i}^{(k)} = \lambda_1^k + \cdots + \lambda_n^k$$

Krilov

$$b_i^{(n-1)}p_1 + b_i^{(n-2)}p_2 + \cdots + b_i^{(0)}p_n = -b_i^{(n)}, \quad i = 1,\ldots,n,$$

$$\mathbf{b}^{(k)} = (b_1^{(k)}, b_2^{(k)}, \ldots, b_n^{(k)})^\top = A\mathbf{b}^{(k-1)} = A^k\mathbf{b}^{(0)}, \qquad \mathbf{b}^{(0)} \text{ proizvoljno}$$

Danilevski

$$T^{-1}\,A\,T = P = \begin{pmatrix} p_1 & \cdots & p_{n-1} & p_n \\ 1 & & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix}, \qquad \begin{matrix} P\,\mathbf{y}_k = \lambda_k \mathbf{y}_k, \\ \mathbf{x}_k = T\,\mathbf{y}_k, \end{matrix} \qquad k = 1,\ldots,n,$$

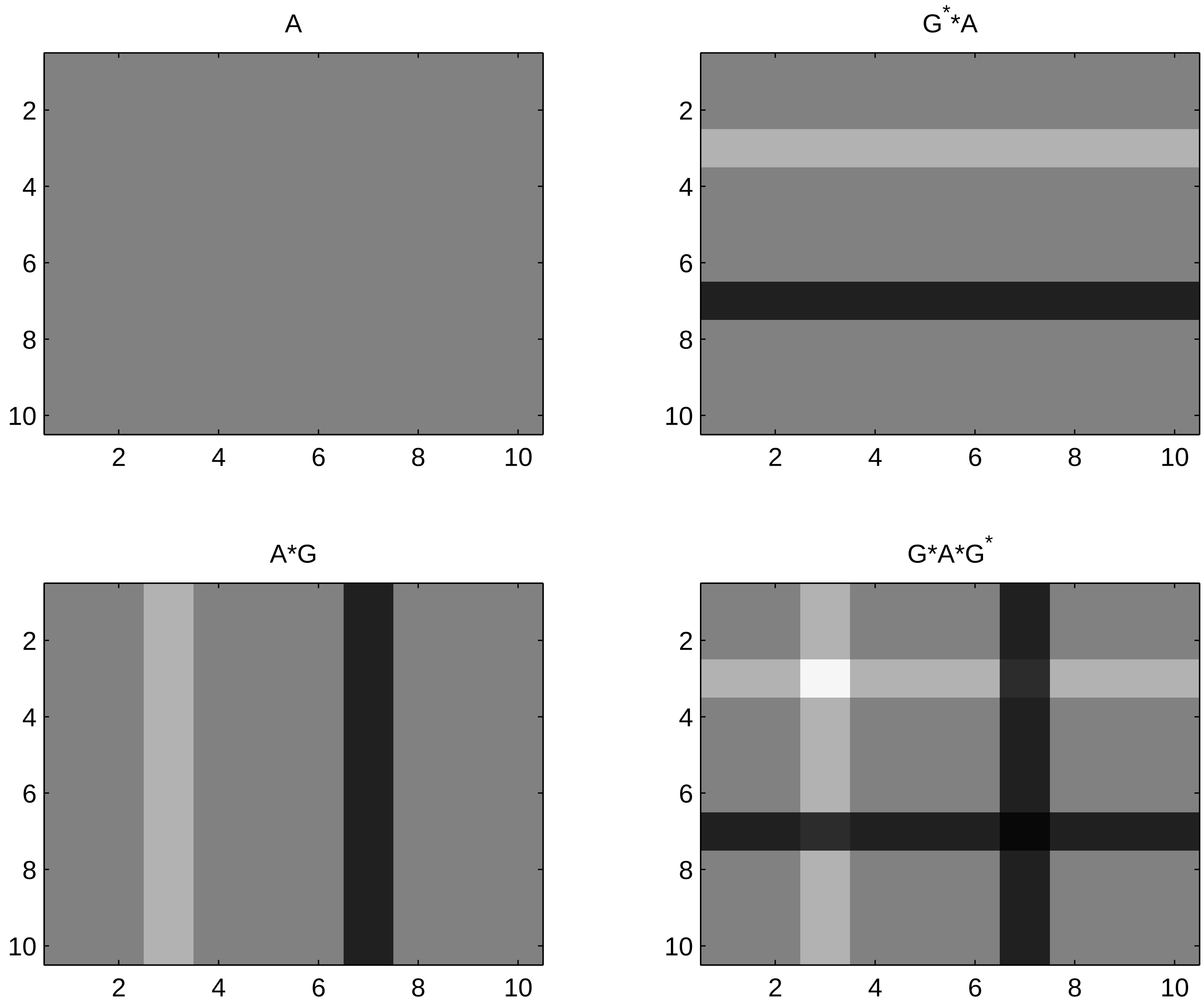
Givens-ova metoda rotacije

(U matrica rotacije, $U^{-1} = U^*$)

$$A \sim A' = U_{n-1,n}^* \cdots U_{24}^* U_{23}^* \, A \, U_{23} U_{24} \cdots U_{n-1,n} = \begin{pmatrix} * & \cdots & \cdots & * \\ * & & & \vdots \\ & \ddots & & \vdots \\ 0 & & * & * \end{pmatrix}$$

$$B = U_{k,l}^* \, A \, U_{k,l}, \qquad U_{kl} = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & \cos\phi & & -e^{-\imath\psi}\sin\phi & & \\ & & & \ddots & & & \\ & & e^{\imath\psi}\sin\phi & & \cos\phi & & \\ & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow k \\ \\ \leftarrow l \\ \\ \\ \end{matrix}$$

$$\cos\phi = \frac{|a_{k,k-1}|}{\sqrt{|a_{k,k-1}|^2 + |a_{l,k-1}|^2}}, \qquad e^{-\imath\psi}\sin\phi = \frac{a_{l,k-1}}{a_{k,k-1}}\alpha \qquad \longrightarrow \qquad b_{l,k-1} = 0$$


A
G**A
A*G
G*A*G*
2
4
6
8
10

Jacobijeva metoda

$$A^* = A$$

$$A_m = U_m^\top \cdots U_1^\top A\, U_1 \cdots U_m \qquad \lim_{m\to\infty} A_m = D = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$$

$$\lim_{m\to\infty} U_1 \cdots U_m = U = \big(\mathbf{x}_1 \quad \ldots \mathbf{x}_n\big)$$

$$U_m \equiv U_{kl} = \begin{pmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & \cos\phi & & -\sin\phi & \\ & & & \ddots & & \\ & & \sin\phi & & \cos\phi & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \leftarrow k \\ \\ \leftarrow l \\ \\ \\ \end{matrix} \qquad \tan 2\phi = \frac{2a_{kl}}{a_{kk} - a_{ll}} \quad \longrightarrow \quad a_{k,l}^{(m)} = 0$$

$$A_m = \{a_{i,j}^{(m)}\}, \quad \nu_m^2 = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} {a_{i,j}^{(m)}}^2, \quad \nu^2 = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} a_{i,j}^2, \quad \sigma \geq n, \qquad \nu_m^2 \leq \nu^2 \Big(1 - \frac{2}{\sigma^2}\Big)^m \underset{m\to\infty}{\longrightarrow} 0$$

tačnost $\quad \nu_m \leq \varepsilon\nu \quad$ za $\quad a_{i,j}^{(m)} \leq \frac{\varepsilon}{n}\,\nu \quad$ ili $\quad m \geq \dfrac{2\ln\varepsilon}{\ln\,(1 - \frac{2}{\sigma^2})}$

Householder-ova metoda

$$A \sim A' = Q^* A\, Q = T_{n-2} \dots T_1\, A\, T_1 \dots T_{n-2} = \begin{pmatrix} * & \cdots & \cdots & * \\ * & & & \vdots \\ & \ddots & & \vdots \\ 0 & & * & * \end{pmatrix}$$

$$T_m = \left(\begin{array}{ccc|ccc} 1 & \dots & 0 & 0 & \ddots & 0 \\ & \dots & & & \dots & \\ 0 & \dots & 1 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & & & \\ \vdots & \ddots & \vdots & & H_m & \\ 0 & \dots & 0 & & & \end{array}\right), \qquad Q = T_1 \dots T_{n-2}$$

$$M = \mathrm{dim}(H_m) = n - m, \quad \mathbf{x}_m = \left(a_{m+1,m}^{(m-1)} \quad \dots \quad a_{n,m}^{(m-1)}\right)^\top \quad \left(A_m = \left\{a_{i,j}^{(m)}\right\} = T_m\, A_{m-1},\ A_0 \equiv A\right)$$

$$\mathbf{x} = (x_1, \dots, x_M)^\top, \quad \sigma = \sqrt{\sum_{i=1}^{M} |x_i|^2}, \quad x_1 = |x_1| e^{\imath\alpha}, \quad k = -\sigma e^{\imath\alpha}, \quad \beta = \big(\sigma(\sigma + |x_1|)\big)^{-1},$$

$$\mathbf{u} = \mathbf{x} - k\mathbf{e}_1, \qquad H = I - \beta \mathbf{u}\,\mathbf{u}^* \qquad \left(H = H^* = H^{-1}\right)$$

LR metoda

$$A_i = L_i\, R_i, \qquad L_i = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ * & \dots & 1 \end{pmatrix}, \quad R_i = \begin{pmatrix} * & \dots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

$$A_{i+1} = R_i\, L_i = L_{i+1}\, R_{i+1}, \qquad i = 1, 2, \dots .$$

$$A_{i+1} = L_i^{-1}\, A_i\, L_i = (L_1 \cdots L_i)^{-1}\, A_1\, (L_1 \cdots L_i), \qquad A^i \equiv A_1^i = T_i\, U_i = (L_1 \cdots L_i)\,(R_i \cdots R_1)$$

Teorema: $\lim_{i\to\infty} A_i = \lim_{i\to\infty} R_i = \begin{pmatrix} \lambda_1 & \dots & * \\ & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}, \qquad \lim_{i\to\infty} L_i = I$ ako

- $L\,R$ dekompozicije $A_i = L_i\, R_i$ postoje za svako $i = 1, 2, \dots,$
- $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|,$
- postoje dekompozicije $X = L_x\, R_x, \qquad Y = L_y\, R_y,$ gde je

$$A = X\, D\, Y, \qquad D = \text{diag}(\lambda_1, \dots, \lambda_n), \qquad X = \left(\mathbf{x}_1, \dots, \mathbf{x}_n\right)^\top, \qquad Y = X^{-1}$$

Q R metoda $A_i = Q_i\, R_i, \qquad Q_i^{-1} = Q_i^*, \qquad R_i = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$

$$A_{i+1} = R_i\, Q_i = Q_{i+1}\, R_{i+1}, \qquad i = 1, 2, \ldots .$$

$$A_{i+1} = Q_i^*\, A_i\, Q_i = (Q_1 \cdots Q_i)^*\, A_1\, (Q_1 \cdots Q_i), \qquad A^i \equiv A_1^i = P_i\, U_i = (Q_1 \cdots Q_i)\,(R_i \cdots R_1)$$

Teorema:

$$\lim_{k\to\infty} S_{k-1}^*\, Q_k\, S_k = I, \qquad \lim_{k\to\infty} S_{k-1}^*\, A_k\, S_{k-1} = \lim_{k\to\infty} S_k^*\, R_k\, S_{k-1} = \begin{pmatrix} \lambda_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}$$

$$S_k = \text{diag}\big(e^{\imath\phi_1^{(k)}}, \ldots, e^{\imath\phi_n^{(k)}}\big), \qquad \lim_{k\to\infty} a_{jj}^{(k)} = \lambda_j, \quad j = 1, \ldots, n, \qquad \Big(A_k = \{a_{jj}^{(k)}\}\Big)$$

ako je $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ i postoji dekompozicija $Y = L_y\, R_y$,

$$A = X\, D\, Y, \quad D = \text{diag}(\lambda_1, \ldots, \lambda_n), \quad Y = X^{-1}, \quad L_y = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ * & \cdots & 1 \end{pmatrix}, \quad R_y = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}$$

Delimičan problem sopstvenih vrednosti $\boxed{\lambda_1,\, \mathbf{x}_1}$

$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$

Metoda proizvoljnog vektora

$$\lim_{k\to\infty} \frac{v_j^{(k+1)}}{v_j^{(k)}} = \lambda_1, \qquad \lim_{k\to\infty} \mathbf{v}^{(k)} = \mathbf{x}_1$$

$\mathbf{v}^{(k)} = \left(v_1^{(k)} \quad \ldots \quad v_n^{(k)}\right)^\top, \qquad \mathbf{v}^{(k)} = A\,\mathbf{v}^{(k-1)} = A^k\,\mathbf{v}^{(0)}, \qquad \mathbf{v}^{(0)}$ proizvoljno

Metoda skalarnog proizvoda

$$\lim_{k\to\infty} \frac{\left(\mathbf{v}^{(k)},\, \mathbf{w}^{(k)}\right)}{\left(\mathbf{v}^{(k-1)},\, \mathbf{w}^{(k)}\right)} = \lambda_1, \qquad \lim_{k\to\infty} \mathbf{v}^{(k)} = \mathbf{x}_1$$

$\mathbf{v}^{(k)} = A\,\mathbf{v}^{(k-1)} = A^k\,\mathbf{v}^{(0)}, \qquad \mathbf{w}^{(k)} = A^*\mathbf{w}^{(k-1)} = (A^*)^k\,\mathbf{w}^{(0)}, \qquad \mathbf{v}^{(0)},\, \mathbf{w}^{(0)}$ proizvoljno

Metoda tragova

$$|\lambda_1| = \lim_{k\to\infty} \sqrt[k]{|\mathrm{tr}(A^k)|}, \qquad \lambda_1 = \lim_{k\to\infty} \frac{\mathrm{tr}(A^{k+1})}{\mathrm{tr}(A^k)}, \qquad \mathrm{tr}(A^k) = \sum_{i=1}^{n} a_{i,i}^{(k)}$$

$$\mathbf{x}_1 \approx A^k\,\mathbf{v}^{(0)}, \qquad \mathbf{v}^{(0)} \text{ proizvoljno}$$

Metoda iscrpljivanja

$$A_1 = A - \lambda_1 \mathbf{x}_1\,\mathbf{y}_1^*$$

$$A_1\,\mathbf{x}_1 = \mathbf{0}, \quad A_1\,\mathbf{x}_k = \lambda_k\,\mathbf{x}_k, \quad k = 2, \ldots, n.$$

$$A^*\,\mathbf{y}_k = \overline{\lambda}_k\,\mathbf{y}_k, \qquad \big(\mathbf{x}_1,\,\mathbf{y}_1\big) = \mathbf{y}_1^*\,\mathbf{x}_1 = 1$$